7B Math Review: Taylor Series

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January 23, 2018

Taylor Series

Definition

The Taylor series of a function $f(x)$ at the point $x = a$ is defined as

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + ...$$

The intuition behind this definition is that at the point $x = a$, all the derivatives of the expansion match the function exactly. When you take the nth derivative, all the smaller derivatives have been "killed" already by differentiation, and all the higher derivatives will disappear because they all have a term with $(x-a)$ that is zero, since we're at $x = a$. (And the factorials in the front of the surviving term will disappear, as you "pull down" powers from differentiating, so you have exactly the right answer!)

We often use Taylor series to make approximations when we do physics. Often, the real function is complicated, so we just take the important terms, and forget about the details (the physics jargon is "higher order terms"). In 7B, we will only consider first, and sometimes second order terms as important (that is, terms that are linear or quadratic in x.)

Very important thing to note: this approximation is only valid near $x = a$!!! You can't apply a Taylor expansion at $x = a$ to some other point $x = b$ if b is not close to a! What "close to" means, exactly, is another story... knowing is a bit of an art, but scientists are artists, after all.

Practice Questions

1. Q: Compute the Taylor series of $f(x) = \frac{1}{(1-x)^2}$, to first order, about the point $x = 0$.

   A: $f'(x) = -2(1-x)^{-3} \cdot -1 = 2(1-x)^{-3}$.

   $\implies f(x) \approx 1 + 2x$, around $x = 0$

2. Q: Compute the first two non-zero terms of the Taylor series of $f(x) = \sin(x)$ about $x = 0$.

   A: $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$.

   $\implies f(x) \approx x - \frac{1}{3!}x^3$, around $x = 0$

Other good expansions to know are expansions for $\cos(x)$ and e^x. Try to do them!

Practical Example

Recall the lecture where we derived the formula for the height of the tide h, and made the approximation that

$$\frac{1}{(a-R)^2} = \frac{1}{a^2}\frac{1}{(1-R/a)^2} \approx \frac{1}{a^2}\Big(1 + \frac{2R}{a}\Big).$$

In this case, since $R << a$, that means R/a is a small, small number- close to zero! So we are in the regime where we can apply the Taylor approximation.