Astronomy 3
Problem Set 3 Solutions

Problem 1: You can estimate the lifetime of the Sun using simple math. The total mass of the Sun is about $2 \cdot 10^{30}\ kg$, of which about 75% was hydrogen when the Sun formed. Only about 13% of this hydrogen is available for thermonuclear fusion; the rest remains in the outer layers of the Sun.
(1) Based on the given information, calculate the total mass of hydrogen available for fusion over the lifetime of the Sun.
(2) How much energy will have been released by the Sun when the hydrogen in part (1) is consumed?
(3) The Suns luminosity is $4 \cdot 10^{26}\ Watts$, where 1 Watt is equal to 1 Joule per second. At this rate, how many seconds would it take to consume the hydrogen that you calculated above? How many billion years is it?
(4) How old is the Sun now? How many more years will it remain as a main sequence (i.e.hydrogen-burning) star?

Solution

75% of $2 \cdot 10^{30}\ kg$ is hydrogen. 13% of that 75% of $2 \cdot 10^{30}\ kg$ is available for fusion.

$$0.13 \cdot 0.75 \cdot 2 \cdot 10^{30}\ kg = 0.195 \cdot 10^{30}\ kg$$

$$\boldsymbol{M_H = 1.95 \cdot 10^{29}\ kg}$$

That hydrogen is available for fusion. Over the lifetime of the Sun, it will be converted into helium. Not all of that *mass* will become helium though. 0.7% of it will be burned off as energy.

$$lost\ mass = 0.007 \cdot 1.95 \cdot 10^{29}\ kg = 1.365 \cdot 10^{27}\ kg$$

This lost mass is converted into energy according to the famous relation $E = mc^2$.

$$E = mc^2 = (1.365 \cdot 10^{27}\ kg)(3 \cdot 10^8\ m/s)^2 = 1.365 \cdot 10^{27} \cdot 9 \cdot 10^{16}\ kg\ m^2\ s^{-2}$$

The units $kg\ m^2\ s^2$ are equivalent to the SI unit of energy, the joule (J).

$$\boldsymbol{E \approx 1.23 \cdot 10^{44}\ J}$$

The Sun will burn through this at a rate of $4 \cdot 10^{26}\ W$.

$$time = \frac{Energy}{Power} = \frac{1.23 \cdot 10^{44}\ J}{4 \cdot 10^{26}\ W} = 3.075 \cdot 10^{17}\ s$$

$$\boldsymbol{t_\odot \approx 3.08 \cdot 10^{17}\ s \approx 9.77\ billion\ years}$$

The Sun, according to the Internet, is around 4.5 billion years old right now. Rounding our answer to 10 billion years, the Sun has about 5.5 billion years left.

Problem 2: White dwarfs are dense. Sirius B is the first white dwarf ever discovered. It is the binary companion star of Sirius A, the brightest star in the sky. The surface temperature of Sirius B is about 5 times that of the Sun, but its luminosity is only 0.0025 times that of the Sun, and its radius is 0.01 times that of the Sun.
(1) The mass of Sirius B is very similar to that of the Sun. The Suns average density is 1.5 g/cm^3. What is the average density of Sirius B? (Hint: Recall density is equal to mass divided by volume, and volume is proportional to R^3, where R is the radius.)
(2) Calculate the mass of a teaspoonful of the Sun in kilograms. (Assume a teaspoon holds a volume of 1 cm^3.) What about the mass of a teaspoonful of Sirius B in kilograms? How does each of them compare to your (approximate) mass?

Solution

We will assume that Sirius B has the same mass as the Sun. We know density (ρ) is proportional to *mass / volume*, which in turn is proportional to *mass / radius*^{3}. We won't worry about constants here. It doesn't hurt to add in the $\frac{4\pi}{3}$ component of volume, but it doesn't make much of a difference on an astronomical scale (and the answer looks nicer). The Sun's density is given by:

$$\rho_\odot \propto \frac{M_\odot}{R_\odot^3}$$

Substituting in what we know about Sirius B, its density is given by:

$$\rho_{SB} \propto \frac{M_{SB}}{R_{SB}^3} \propto \frac{M_\odot}{(0.01R_\odot)^3} \propto \frac{M_\odot}{(10^{-2} \cdot R_\odot)^3} \propto \frac{M_\odot}{10^{-6}\ R_\odot^3} \propto 10^6 \cdot \frac{M_\odot}{R_\odot^3}$$

We know that $M_\odot / R_\odot^3 \propto \rho_\odot$, so we can substitute that back into the Sirius B relation:

$$\rho_{SB} \propto 10^6 \cdot \rho_\odot$$

Substituting in the numerical value for $\rho_\odot$:

$$\boldsymbol{\rho_{SB} \approx 1.5 \cdot 10^6\ g/cm^3}$$

If a teaspoon is about 1 cm^3, then one teaspoon of the Sun is:

$$M_{tsp,\odot} = \rho_\odot \cdot 1\ cm^3 = 1.5\ g/cm^3 \cdot 1\ cm^3 = \boldsymbol{1.5\ g}$$

A teaspoon of the Sun weighs little more than a paperclip. How about Sirius B?

$$M_{tsp,SB} = \rho_{SB} \cdot 1\ cm^3 = 1.5 \cdot 10^6\ g/cm^3 \cdot 1\ cm^3 = \boldsymbol{1.5 \cdot 10^6\ g}$$

A teaspoon of Sirius B weighs in at 1500 kg; that's something like 21x the average human body mass of 70 kg.

Problem 3: Neutron stars are denser. A neutron star has a much higher density than a white dwarf: about $4 \cdot 10^{14}\ g/cm^3$!
(1) Again, suppose a teaspoon holds a volume of 1 cm^3. What is the mass of a teaspoonful of a neutron star in kilograms? Calculate how many people of your (approximate) mass must stand on a scale in order to balance a teaspoonful of a neutron star. How does it compare to the world population? (Are you impressed?)
(2) Using the density given above and the equation that relates mass, density, and radius of a spherical object, calculate the radius in kilometers of a 1.5 M neutron star. (Use $1\ M_\odot = 2 \cdot 10^{33}\ g$). How does this radius compare to the approximate size of the City of Berkeley?

Solution

This conversion is pretty straightforward:

$$M_{tsp,NS} = 4 \cdot 10^{14}\ g/cm^3 \cdot 10^{-3}\ kg/g \cdot 1\ cm^3 = \boldsymbol{4 \cdot 10^{11}\ kg}$$

An average human body mass is 70 kg.

$$N = \frac{4 \cdot 10^{11}\ kg}{70\ kg/person} \approx 5.7 \cdot 10^9\ people$$

A teaspoon of a neutron star weighs as much as almost 6 billion people: almost the entire human population. You should be impressed.
The density-mass-volume relation is:

$$\rho = \frac{M}{V}$$

Assuming a spherical neutron star, $V = \frac{4\pi}{3}R^3$.

$$\rho = \frac{M}{\frac{4\pi}{3}R^3}$$

Solving for R:

$$R = \left[\frac{3M}{4\pi\rho}\right]^{\frac{1}{3}} = \left[\frac{3 \cdot 1.5 \cdot 2 \cdot 10^{33}\ g}{4 \cdot \pi \cdot 4 \cdot 10^{14}\ g/cm^3}\right]^{\frac{1}{3}} = \left[\frac{9}{16\pi} \cdot 10^{33-14}\ cm^3\right]^{\frac{1}{3}} = (1.79 \cdot 10^{18}\ cm^3)^{\frac{1}{3}}$$

$$R \approx 1.2 \cdot 10^6\ cm$$

$$\boldsymbol{R \approx 12\ km}$$

Berkeley has a radius of something like 2 or 3 km, so a neutron star is only 4 to 6 times the size of Berkeley. Of course, you might measure the entire city to be 6 km across, or you might measure campus to be something like 1 km; it doesn't really matter. The idea is that a neutron star with a mass of 1.5 $M_\odot$ is about the size of a small city.