Astro 7B – Problem Set 10

1 Old School Universe

Taken from Ryden and Peterson 23.4.

In the good old days before we had any data, many of us imagined ourselves to live in a "Newtonian" universe: one filled only with ordinary matter, and one which is spatially flat (Euclidean, where triangles look like triangles and all the rules of elementary school geometry are valid). A flat universe is one whose average density ρ (averaged over cosmologically large volumes, i.e., Gpc^3) equals the *critical density* ρ_c. In a flat universe, the "scale factor" $a(t)$ obeys:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho_c}{3} \tag{1}$$

The scale factor is the "Adobe Illustrator" expansion parameter for space. It measures how big the universe is at any given time. By convention, we set $a(t = t_{\mathrm{now}}) = 1$ at the current time. Thus, in the past, $a(t < t_{\mathrm{now}}) < 1$; and in the future, $a(t > t_{\mathrm{now}}) > 1$ (assuming an expanding universe).

Note that the critical density ρ_c changes with time. In a flat universe, at any given time, the actual density just follows the critical density: $\rho(t) = \rho_c(t)$.

(a) Assume that at $t = t_{\mathrm{now}}$, $\rho_c = \rho_{c,\mathrm{now}}$. **Write down an expression for $\rho(t) = \rho_c(t)$ in terms of $\rho_{c,\mathrm{now}}$ and $a(t)$.** Remember that for our Newtonian universe, there is only ordinary matter. Think about how the density of ordinary matter changes as space expands. You might find aspects of a previous problem set on "stretchy photons" helpful.

(b) Insert your answer for (a) into equation (1) and **solve for $a(t)$.**

Your answer for $a(t)$ should depend only on t and t_{now}. Use the boundary conditions $a(t_{\mathrm{now}}) = 1$ and $a(t = 0) = 0$ to substitute away $G\rho_{c,\mathrm{now}}$ from your answer.

(c) **What is t_{now} in terms of the Hubble constant now, H_{now}?** Use the fact that $H(t) \equiv \dot{a}(t)/a(t)$.

(d) Today we know that (i) $H_{\mathrm{now}} = 70$ km/s/Mpc and (ii) the oldest stars in the universe have an age of $t_* = 13$ Gyr. **Explain why these two observations are or are not consistent with a flat Newtonian universe (remember, the universe must be at least as old as the things that are in it)**.

2 Taylor Expansions of Space

In class we expanded the scale factor (a.k.a. the cosmological expansion parameter) $a(t)$ as a power series:

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a}(t_0)(t - t_0)^2 + ... \tag{2}$$

We rewrote this using standard notation, defining the Hubble constant $H_0 \equiv \dot{a}(t_0)/a(t_0)$ and the deceleration parameter $q_0 \equiv -\ddot{a}(t_0)/[a(t_0)H_0^2]$:

$$a(t) = 1 + H_0(t - t_0) - \frac{q_0}{2}\left[H_0(t - t_0)\right]^2 + ... \tag{3}$$

In everything that follows in this problem, we will assume that

$$\epsilon(t) \equiv H_0(t - t_0) \ll 1\,. \tag{4}$$

This assumption says that we are considering only times in the not-too-distant past; i.e., time intervals $(t - t_0)$ so short that space expands just a bit. In other words, our expressions will be valid only for the low-redshift ($z \ll 1$ = local = nearby) universe. However, at the end of this problem, we will abuse our derivation and cavalierly use our expressions at $z \sim 1$; this is quantitatively inaccurate but is good enough to get a qualitative feeling for how the redshift-magnitude plot should look.

Equation (3) rewritten with (4) equals:

$$a(t) = 1 + \epsilon - \frac{q_0}{2}\epsilon^2 + ... \tag{5}$$

(a) In class we derived an expression for the co-moving coordinate r of an object (assuming that $r = 0$ corresponds to you = the observer):

$$r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} \tag{6}$$

This is the co-moving coordinate of an object (read: supernova) which emitted a photon at time t_e (subscript e for "emit") — a photon that was later detected by you at time t_0 (subscript 0 = "nought" = "now").

Insert (5) into (6), and use (4) to derive:

$$r = c(t_0 - t_e) + \frac{1}{2}cH_0(t_0 - t_e)^2 + ... \tag{7}$$

Important: You are going to have to Taylor expand.

(b) **Express $a(t_e)/a(t_o)$ in terms of the redshift z of the emitting object. Your expression should be exact, given the formula presented in class (and in either textbook).**

(c) **Taylor expand your answer in (b) in powers of z. Then combine with (5) to find:**

$$\epsilon(t_e) = -z + (1 + q_0/2)z^2 + ... \tag{8}$$

Important: when Taylor-expanding, keep terms of order z^2.

Depending on how you solve it, you may be dealing with a quadratic at some point; if so, your solution for $\epsilon(t_e)$ will have two roots. Decide the sign of $\epsilon(t_e)$ and thus the appropriate root.

(d) **Insert (8) into (7) to find:**

$$r = cH_0^{-1}\left[z - \frac{1}{2}z^2(1 + q_0)\right] \tag{9}$$

(e) In class we derived the "luminosity distance" as the distance inferred from a standard candle:

$$d_L = a(t_0)r(1 + z) \tag{10}$$

Use (d) to find:

$$d_L = cH_0^{-1}\left[z + \frac{1}{2}z^2(1 - q_0)\right] \tag{11}$$

(f) Recall how "apparent magnitude" m is a measure of flux F:

$$m = -2.5\log_{10}\left(\frac{F}{F_{\rm ref}}\right) \tag{12}$$

where $F_{\rm ref}$ is some internationally agreed-upon reference flux.

Recall also how "absolute magnitude" M is a measure of luminosity:

$$M = -2.5\log_{10}\left(\frac{F_{10}}{F_{\rm ref}}\right) \tag{13}$$

where F_{10} is the flux that the object *would have IF it were a luminosity distance of* 10 *pc away from the observer.*

The source (read: supernova) actually has a luminosity distance of d_L. **Express F_{10} in terms of F and d_L. Thereby show that:**

$$m - M = 2.5 \log_{10} \left(\frac{d_L}{10\,\mathrm{pc}} \right)^2 \tag{14}$$

The quantity $m - M$ (apparent minus absolute magnitude) is called the "distance modulus". It is a (perverse) measure of distance (actually, luminosity distance).

(g) Combine (14) and (11) to find:

$$m - M = 5 \log_{10} \left[\frac{cH_0^{-1}[z + z^2(1 - q_0)/2]}{10\,\mathrm{pc}} \right] \tag{15}$$

Take $H_0 = 70$ km/s/Mpc. **Plot $m - M$ versus z, for z between 0.01 and 1, for 3 sample values of $q_0 \in (1/2, 0, -1)$.** Your plot should have 3 curves on it corresponding to the 3 example values of q_0. Optional: you can compare your figure to Figures 24.5 and 24.6 of Ryden & Peterson, or Figure 29.26 or 29.27 of Carroll & Ostlie.

(Congratulations — now all you need are some real data to overlay on your plot to see which curve the data best matches, and you can reproduce a Nobel-Prize-winning result.)