Astro 7B – Problem Set 11

1 The End of the Observable Universe (or The Beginning of Astronomy)

There is a limit to how far electromagnetic radiation can reach us from the ends of the universe. That limit is set by the moment of recombination: the moment when electrons and protons first united to become neutral hydrogen atoms. After recombination,[1] photons are suddenly free to travel across the universe. Before recombination, photons cannot travel far because they are too busy scattering off free electrons (via Thomson scattering, a.k.a. Compton scattering). After recombination, the photons free-stream in all directions (except the small fraction of photons that happen to have wavelengths that match the line transitions in neutral hydrogen). We detect this radiation today as the cosmic microwave background (CMB) radiation.

We cannot detect electromagnetic radiation from before the moment of recombination. Trying to see beyond the moment of recombination is tantamount to seeing through an optically thick cloud of free electrons. Before recombination, the photons from the Big Bang are trapped in a dense thicket of electrons. After recombination, the photons are free.

Recombination is also called the moment when radiation "de-couples" from matter. Recombination might also be called the moment when conventional astronomy — the study of light from distant portions of the universe — begins.

In the early universe, matter is squeezed to high density, and frequent collisions between particles ensure that everything relaxes into thermal equilibrium at a common temperature T.

Early on, T is so high that almost all hydrogen is ionized. As the universe expands, T falls. When T is low enough, hydrogen becomes neutral. This is the moment of recombination.

(a) Assume that today ($t = t_0$) the total baryon (read: hydrogen) density is $n_0 \sim 10^{-7}$ cm^{-3} — this is a gross average over cosmologically large volumes. **Write down an exact expression for the total baryon density $n(t)$ as a function of n_0 and the scale factor $a(t)$, assuming $a(t_0) = 1$.**

(b) The temperature today of the cosmic microwave background is $T_0 = 2.73$ K. **Write down an exact expression for the temperature $T(t)$ of the radiation from the big**

[1] "Recombination" is more properly called "combination" since this is the first time electrons and protons combine.

bang in terms of T_0 and the scale factor $a(t)$ (You may find a previous problem set helpful).

(c) In thermal equilibrium, the number density of free protons n_p, the number density of free electrons n_e, and the number density of neutral hydrogen atoms n_H are given by the Saha equation:[2]

$$\frac{n_p n_e}{n_H} \sim \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT} \tag{1}$$

Here m_e is the mass of an electron, k is Boltzmann's constant, h is Planck's constant, and $\chi = 13.6$ eV is the ionization energy for (ground-state) hydrogen.

Assume that the only form of baryonic matter in the universe is hydrogen.[3] Define the moment of recombination as the moment when $n_p = n_H$ (half the hydrogen is neutral and the other half is ionized).

Use (a), (b), and the Saha equation to calculate the scale factor $a(t_{rec})$ and temperature $T(t_{\rm rec})$ at the moment of recombination.

(d) Assume that from $t_{\rm rec}$ to t_0, the universe is flat and matter-dominated.[4] **For such a universe, write down an exact expression for $a(t)$ in terms of t and t_0 (you may find a previous problem set helpful). Then combine with (c) to calculate $t_{\rm rec}$ in terms of t_0. Finally convert to years knowing that $t_0 = 13.8$ Gyr.** This is how old the universe was at the time of recombination.

2 How to Get to Carnegie Hall (or Practicing Your Scalings)

The universe divides into (at least) 5 eras: in chronological order: radiation-dominated, inflation-dominated, radiation-dominated, matter-dominated, and dark-energy-dominated. In each era, assume that only one kind of energy exists, so that the scale factor $a(t)$ behaves either as a pure power law or as a pure exponential with time. Approximating $a(t)$ as a piecewise function is the same approximation we used in class.

[2] People who are familiar with the Saha equation will appreciate that for this problem we have approximated the annoying order-unity ratio of partition functions as 1; hence the use of $\sim$.

[3] This is not a terrible approximation. By number, hydrogen dominates all other elements by roughly a factor of 10.

[4] This is also not a terrible approximation. As we discussed in class, only in recent times (times near t_0) has dark energy become competitive with (dark) matter in terms of the energy density budget.

This problem gives practice in scalings and in calculating proper distances and horizon distances. It DOES NOT make the assumption made in class that the current time t_0 equals the time when the energy density of matter equals the energy density of dark energy. That is, we distinguish now between t_0 and $t_{\mathrm{m}\Lambda}$, where $t_{\mathrm{m}\Lambda}$ equals the time when the energy density of matter equals the energy density of dark energy.

Unless otherwise noted, all the notation in this problem is identical to the notation used in lecture.

(a) Write down formulae for the scale factor $a(t)$ for each era:

$t_{\mathrm{m}\Lambda} < t < t_0$: $a(t) \propto \exp[(8\pi G u_\Lambda/3c^2)^{1/2} t]$

$t_{\mathrm{rm}} < t < t_{\mathrm{m}\Lambda}$: $a(t) \propto t^{2/3}$

$t_{\mathrm{f}} < t < t_{\mathrm{rm}}$: $a(t) \propto t^{1/2}$

$t_{\mathrm{i}} < t < t_{\mathrm{f}}$: $a(t) \propto \exp[(8\pi G u_{\mathrm{inflation}}/3c^2)^{1/2} t]$

$0 < t < t_{\mathrm{i}}$: $a(t) \propto t^{1/2}$

In other words, find the coefficients to convert the above proportionalities into equalities. Express your answers in terms of $a(t_0)$, $u_\Lambda =$ constant, $u_{\mathrm{inflation}} =$ constant, and the various t's.

Hint: $a(t = 0) = 0$.

(b) At t_0, astronomers measure $\Omega_{\Lambda,0} \equiv u_{\Lambda,0}/u_{\mathrm{crit},0} = 0.74$ and $\Omega_{\mathrm{m},0} \equiv u_{\mathrm{m},0}/u_{\mathrm{crit},0} = 0.26$. Find $a(t_{\mathrm{m}\Lambda})$, i.e., the scale factor at matter-dark-energy equality, when $u_\Lambda = u_{\mathrm{m}}$.

(c) Use your answers in (a) and (b) to solve for $t_{\mathrm{m}\Lambda}$, the time of matter/dark-energy equality. Use $t_0 = 13.8$ Gyr, $H_0 = 70$ km/s/Mpc, and express your answer for $t_{\mathrm{m}\Lambda}$ in Gyr.

(d) Calculate the horizon distance $d_{\mathrm{hor}}(t_{\mathrm{i}})$ at t_{i} (the horizon distance at the beginning of inflation), assuming $t_{\mathrm{i}} = 10^{-36}$ s. Express in cm.

(e) Calculate the horizon distance $d_{\mathrm{hor}}(t_{\mathrm{f}})$ at t_{f} (the horizon distance just after inflation is over), assuming $N \equiv [8\pi G u_{\mathrm{inflation}}/3c^2]^{1/2}(t_{\mathrm{f}} - t_{\mathrm{i}}) = 70$ (i.e., there were $N = 70$ e-foldings of the universe during inflation) and $t_{\mathrm{f}} = 10^{-34}$ s. Express in cm.

(f) Calculate the proper distance to the surface of recombination, evaluated at t_0. (Since we cannot see photons beyond the surface of recombination because the universe is too optically thick before recombination, we may regard this proper distance as the "size of the visible-light universe"; see previous problem.) Take $t_{\mathrm{rec}} = 3 \times 10^5$ yr and use the fact that recombination occurs after radiation-matter equality but before matter-dark-energy equality. Use whatever information you need in previous parts of this problem, and express in Gpc.

(g) Calculate the proper distance to the surface of recombination, but now evaluated at $t_{\rm f}$. Use $t_{\rm rm} = 10^5$ yr, and whatever information you need from previous parts of this problem. Express your answer in cm, and make sure it is much less than your answer in (e), thereby showing that everything within the visible-light universe was in causal contact thanks to inflation (i.e., the horizon was inflated to a giant size, growing way, way past the surface of recombination).