Astro 7B – Problem Set 9

1 Rotation Curves and Dark Matter Density

Consider a simplified but still semi-realistic rotation curve for the Galaxy: one that is absolutely flat. That is,

$$v_c(r) = v_0 = \text{constant} \tag{1}$$

for ALL r (even the origin).

Assume that this rotation curve results purely from dark matter that is distributed spherically. Assume further that the density profile for the dark matter halo takes the form:

$$\rho(r) = kr^n \tag{2}$$

where k and n are constants. In other words, the halo has a power-law density profile.

(a) **Determine k and n in terms of v_0 and fundamental constants.**

(b) At the Sun's location in the Galaxy, $r = r_0 = 8$ kpc. Also, $v_0 = 250$ km/s. **Use these values to deduce the local density of dark matter in the Solar neighborhood $\rho(r_0)$. Express in units of the g/cm^3.**

(c) Your super-symmetric string theorist friend predicts the mass of the dark matter particle to be about 100 times the mass of a proton. **Estimate from this and your answer in (b) how many dark matter particles are inside (really passing through) your body at any given instant.** (Answers within an order of magnitude of ours will receive full credit.)

2 Standard Rulers and Hubble's Law

Figure 1 shows some classic photographic images of elliptical galaxies. The galaxies are labeled by the names of the clusters in which they reside.

Every image is taken at the same *plate scale* of 7.5 arcsec per millimeter. That is, if you print out Figure 1 on a standard 8.5 inch x 11 inch piece of paper, and if you find that the galaxy measures 1 millimeter across on that piece of paper, then that galaxy subtends 7.5 arcseconds on the sky. (The plate scale is determined entirely the optics of your telescope: specifically, the focal length.)

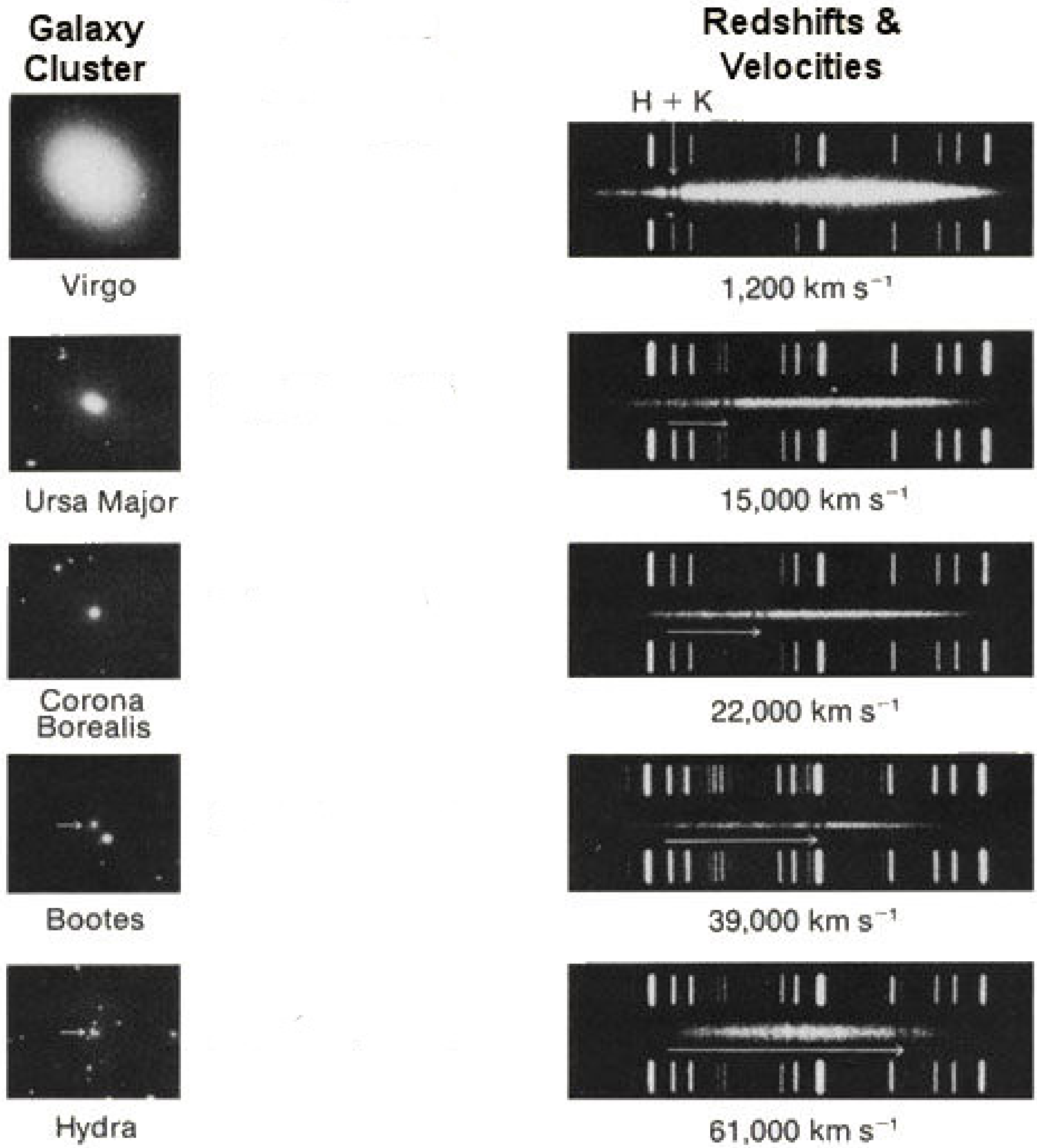

Figure 1: Photographic images of elliptical galaxies, all taken at the same plate scale of 7.5 arcsec per mm (when this figure is printed on a standard 8.5 inch x 11 inch paper).

(a) Assume that every one of the elliptical galaxies shown in Figure 1 represents a *standard ruler*: an object of fixed standard size. Assume that every one of these ellipticals is actually an oblate sphere: a sphere whose polar axis is shorter than its equatorial axis (a sphere that is squashed at the poles). Take the polar axis to have a fixed (standard) length of 15 kpc; this is the distance from the "north pole" of the oblate sphere to the "south pole". Take the equatorial DIAMETER of the oblate sphere to have a fixed (standard) length of 30 kpc. In general, the galaxies have random orientations in space.

All galaxies are optically thin: we can see the light of every star because no star blocks any other. Galaxies only look like solid patches of white (as they do in Figure 1) because our telescopes lack the angular resolution to resolve the tiny spaces (tiny in angle) between the stars. The result is that all the stars blur together.

Make measurements of Figure 1 to estimate the distances d to each of the galaxies in Mpc. You may skip the galaxy in Hydra because the image resolution may be too poor. Answers within a factor of 50% of ours will be given full credit.

(b) Spectra of all the galaxies have been taken. As indicated on Figure 1, the Ca II H + K lines are observed to have significant Doppler shifts relative to their rest wavelengths. The radial velocities (a.k.a. recessional velocities) inferred from these Doppler-shifted lines are listed in Figure 1.

Make a rough plot of radial velocity v_r (in km/s) versus distance d (in Mpc) for the galaxies shown in Figure 1 (again, you may skip Hydra if you wish). Verify that the data approximately conform to a line and measure the best-fit (by eye) slope of this line. Call this slope H_0 and express in km/s/Mpc (congratulations — you have made a measurement of the Hubble constant H_0.) Answers within a factor of 2 of ours will be given full credit.

3 Quasar Redshift

Figure 2 shows the spectrum of a quasar (the super-luminous nucleus of an active galaxy), taken at optical wavelengths.

The spectrum shows 4 obvious emission lines (emitted by hot gas in the vicinity of the accreting supermassive black hole). The 4 wavelengths of the 4 emission lines can be identified with the wavelengths of certain common elements in the universe — appropriately *redshifted*:

$$\text{redshift } z \equiv \frac{\lambda_{\text{observed}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}} \tag{3}$$

where $\lambda_{\text{observed}}$ is the wavelength that is observed (i.e., as shown in Figure 2) and λ_{emitted} is

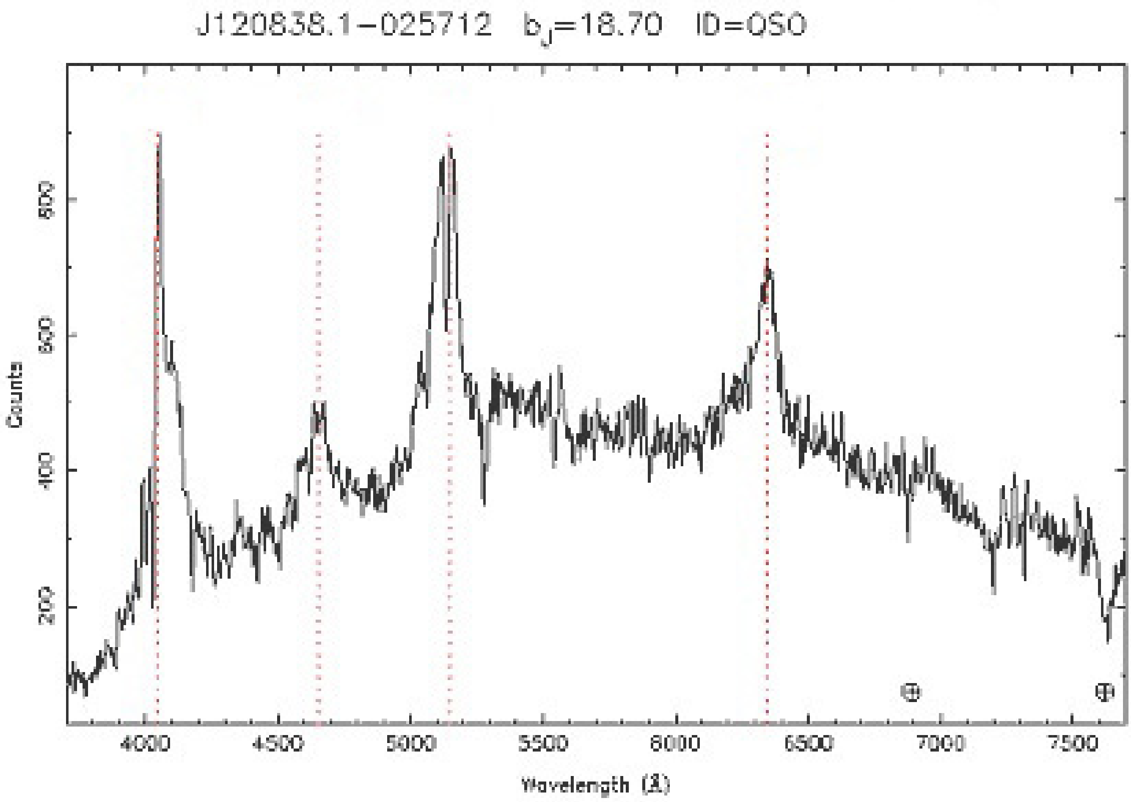

Figure 2: Spectrum of a quasar with 4 obvious emission lines.

the wavelength when the photon was emitted (a.k.a., the "laboratory" wavelength).

What elemental species are responsible for the 4 emission lines shown in Figure 2, and what is the redshift z of this quasar? This involves a guessing game. We take our cue from nearby active galaxies (read: galaxies with actively accreting supermassive black holes) that do not exhibit cosmological redshifts — their redshifts are $z = 0$ — but which commonly show emission lines drawn from the following list:

H Lyman α: $\lambda_{\rm rest} = 1216\mathring{A}$

N V : $\lambda_{\rm rest} = 1241\mathring{A}$.

Si IV : $\lambda_{\rm rest} = 1402\mathring{A}$.

C IV : $\lambda_{\rm rest} = 1549\mathring{A}$.

[C III] : $\lambda_{\rm rest} = 1909\mathring{A}$.

Mg II : $\lambda_{\rm rest} = 2799\mathring{A}$.

[O II]: $\lambda_{\rm rest} = 3728\mathring{A}$.

[Ne III]: $\lambda_{\rm rest} = 3869\mathring{A}$.

H delta: $\lambda_{\rm rest} = 4102\mathring{A}$.

H gamma: $\lambda_{\rm rest} = 4340\mathring{A}$.

[O III]: $\lambda_{\rm rest} = 4363\mathring{A}$.

H beta: $\lambda_{\rm rest} = 4861\mathring{A}$.

[O III]: $\lambda_{\rm rest} = 4956\mathring{A}$.

[O III]: $\lambda_{\rm rest} = 5007\mathring{A}$.

H alpha: $\lambda_{\rm rest} = 6563\mathring{A}$.

Identify the emitting species and the rest wavelengths of the 4 emission lines in Figure 2, and give the redshift z.

Hint: First measure $\lambda_{\rm observed}$ for the 4 lines in the spectrum. Then look for 4 lines in the list above such that $\lambda_{\rm observed}/\lambda_{\rm rest}$ has a nearly constant value.

4 Stretchy Photons

The universe today is filled with *cosmic microwave background (CMB) radiation.* This radiation is observed to be (very nearly) blackbody radiation at a temperature of 2.73 K.

This problem studies how this radiation evolves as the universe expands.

For this problem, you will need to know that the energy density of blackbody radiation is $u = 4\sigma T^4/c$ (where T is the temperature of the radiation). You will also need to know that the pressure of blackbody radiation (actually any form of radiation, as long as it is isotropic) is $P = u/3$.[1]

Consider a volume V that expands along with the universe. Take the *co-moving* volume to be r^3 = constant, so that the *proper* volume $V = [a(t)]^3 r^3$, where $a(t)$ is the usual time-dependent scale factor.

This volume V is filled with photons which, like all many-body systems, obey the first law of thermodynamics:

$$dQ = dE + PdV \tag{4}$$

In a homogeneous, isotropic universe, everything looks the same in space. So the average conditions inside the volume must be the same as the average conditions outside the volume. In particular, the temperature inside the volume must be the same as the temperature outside the volume. This means that there is no heat exchange between the volume and its surroundings: $dQ = 0$ (heat flows from hot to cold, and if there's no temperature gradient, then heats flows nowhere). Therefore, according to the first law, any change in volume dV against pressure P must result directly in a change in energy dE inside the volume (i.e., a change in internal energy).

(a) Sanity check: **if the volume expands, what is the sign of dV? Of dE? Does this accord with your personal experience?**

(b) **Write down the total photon energy E contained inside the volume in terms of T and V.** This part (b) does not involve changes in the volume, so should not involve any integral.

(c) **Use all the relations above to solve for dT/dt in terms of da/dt, a, and T.**

(d) **If $T = T_0$ and $a = 1$ at $t = t_0$, then solve for $T(t)$ in terms of T_0 and $a(t)$.** Your answer should be super-simple — and it should also make sense given (a).

(e) Approximate the blackbody radiation inside the volume as being composed of photons each having the same energy $\sim kT$. **Does the *number* of photons inside the volume change as the universe expands? ALSO, how does the wavelength λ of each of these photons scale with a?**

[1] Recall that pressure and energy density have the same units, a fact that physicists endlessly exploit to avoid having to do hard derivations.