# Astro 250 – Planetary Dynamics – Problem Set 3
# Do at least 1 problem.

Readings: Murray & Dermott 6.1–6.4 (don't let the disturbing expansions disturb you), 6.7-6.9, 6.10.1; ALSO check out the lowest order forms of Lagrange's equations in equations (8.12)–(8.17).

**Problem 1.** The Titan ringlet and the 1:0 Apsidal Resonance

The Colombo ringlet, also known informally as the Titan ringlet, is a narrow planetary ring around Saturn that sits within the 1:0 apsidal resonance established by the largest of the Saturnian moons, Titan. This means that the precession rate of the apsidal line of the ringlet matches the mean motion of Titan; Titan appears to pull the ring along.

Denote Titan's mass over Saturn's mass by $M_T/M = 2.366 \times 10^{-4}$, its semi-major axis by $a_T = 1.22 \times 10^6$ km, its eccentricity $e_T = 0$, its mean motion by $\Omega_T$, and its mean longitude by $\lambda_T$. Denote a single ring particle's semi-major axis by $a = 77871$ km, its eccentricity by $e = 2.6 \times 10^{-4}$, and its mean motion by $\Omega = 2.834 \times 10^{-4}$ rad/s. These numbers are given just for reference; the problem below does not require any numerical evaluation.

a) Write down, to leading order in $e$, the SINGLE term of the disturbing function due to Titan (the perturber) that represents the 1:0 apsidal resonance. Leave all variables in symbolic form (do not plug in numbers).

b) Use Lagrange's equations to compute $\dot{a}$, $\dot{e}$, and $\dot{\tilde{\omega}}$ for the ring particle. Express in terms of the constant $\eta = (M_T/M)\Omega\alpha H_{10}/2$, where $\alpha = a/a_T$ and $H_{10} = 2b_{1/2}^{(1)}(\alpha) + \alpha(d/d\alpha)b_{1/2}^{(1)}(\alpha) - 3\alpha$.

c) The above expression for $\dot{\tilde{\omega}}$ is incomplete because it only accounts for mean-motion resonant forcing by Titan. What is missing is forcing by the secular potential. Let's just say the complete answer is

$$\dot{\tilde{\omega}} = \langle \text{answer in part b} \rangle + \dot{\tilde{\omega}}_{\text{sec}} \tag{1}$$

where $\dot{\tilde{\omega}}_{\text{sec}} = \dot{\tilde{\omega}}_{\text{Saturn}} + \dot{\tilde{\omega}}_{\text{stuff}}$ is the total *additional* precession rate induced by the oblateness of Saturn and the secular potential of everything else—nearby rings (remember the Titan ringlet is just 1 narrow ring embedded in Saturn's gigantic ring complex), Titan, other satellites, the Sun, lost pens, etc. In fact, Saturnian oblateness completely overwhelms the other contributions. There is no need to write out explicitly what all these terms are; we will just work with $\dot{\tilde{\omega}}_{\rm sec}(a)$. (Those of you who did problem 1 of PS 1 know what this function is, but the explicit form is not needed for this problem!)

Further define

$$\epsilon(a) \equiv \dot{\tilde{\omega}}_{\rm sec}(a) - \Omega_T \quad (2)$$

$$\phi \equiv \tilde{\omega} - \lambda_T \quad (3)$$

where $\phi$ is the resonant libration angle. Also define $a = a_0$ such that $\epsilon(a_0) = 0$.

Your equations for $\dot{e}$ and $\dot{\tilde{\omega}}$ are coupled ordinary differential equations. Replace $e$ and $\phi$ in favor of the variables,

$$h \equiv e\cos\phi \quad (4)$$

$$k \equiv e\sin\phi \quad (5)$$

Write down $\dot{h}$ and $\dot{k}$ in terms of $\eta$, $\epsilon$, $h$, and $k$.

d) Solve your equations for $\dot{h}$ and $\dot{k}$. Your solution should contain two arbitrary constants: an amplitude and a phase associated with a sinusoidal oscillation.

e) Plot possible trajectories in $h$ and $k$ space. Identify the conditions under which $\phi$ is circulating (running the gamut from 0 to $2\pi$) or librating (oscillating about a fixed value). If the particle is librating, what are the libration centers, $\langle\phi\rangle$? What libration centers are associated with $a > a_0$? What centers are associated with $a < a_0$? If you have the correct solution, you should notice that something terrible happens at $a = a_0$. This is simply a deficiency of our low-order theory.

## **Problem 2.** Tilted Rings

Consider two planets on circular orbits around a star. The inner planet has mass $m_1$ and semimajor axis $a_1$, and the outer planet has mass $m_2$ and semimajor axis $a_2$. At $t = 0$, the orbit plane of $m_2$ coincides with the $x$-$y$ plane; the orbit plane of $m_1$ is inclined by $i_1$ and has longitude of ascending node $\Omega_1 = 0$ (on the x-axis); and the two planets happen to be passing conjunction at mean longitude $\lambda_1 = \lambda_2 = 0$ (on the x-axis).

a) The period ratio between the two planets $(a_2/a_1)^{3/2}$ is found to be very well approximated by the integer ratio $k : j$, where $k$ and $j$ are relatively prime (only common factor of the two numbers is 1). It is proposed that the two planets occupy a $k : j$ mean-motion resonance.

How many conjunctions occur before the planets pass conjunction again at $\lambda_1 = \lambda_2 = 0$? Argue from this result that when $k \gg j$, the mean-motion resonance is weak.

b) Secular approximation: Smear $m_2$ along its orbit so that it becomes a circular wire of uniform linear mass density. Derive to leading order in $i_1$ the time-averaged disturbing potential felt by $m_1$ due to $m_2$. You will need to perform two integrals: one over the wire that is $m_2$ (integral over $\theta$), and a second over the orbit of $m_1$ (integral over $\psi$). See Figure 1. Use Laplace coefficients (see the integral definition on page 237 of MD).

Hint: Write down $z$ in terms of $a_1$, $i_1$, and $\psi$. Also write down $d$ in terms of $a_1$, $a_2$, $\theta$, and $z$. This $d$ is the denominator of the disturbing potential. Write down the integral over $\theta$ for the potential of the outer wire as evaluated at a single point on the inner orbit. Taylor expand the integrand in the small parameter $z$ BEFORE trying to perform the integral over $\theta$. Your $\theta$-integral should produce Laplace coefficients (following the physicist's maxim that any integral we can't do is given an honorary name). Finally time-average (integrate) over $\psi$.

You can check your answer by looking up the appropriate secular term (the term that does not depend on any mean longitudes) in Appendix B of MD. Note that $s$ in the notation of MD is actually $\sin i/2$.

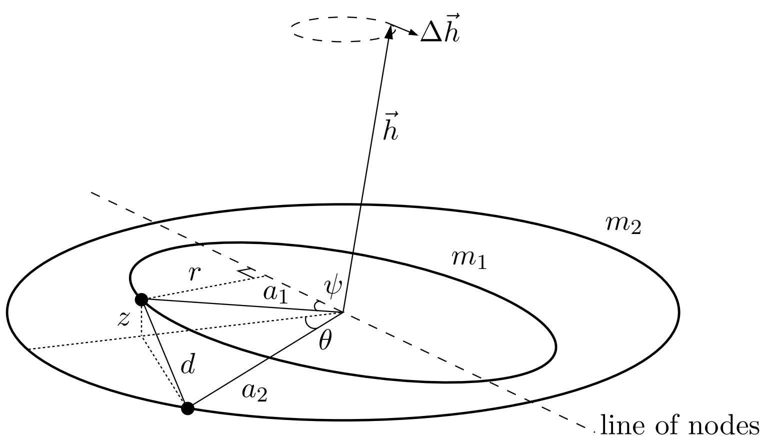


Figure 1: Schematic of tilted rings problem. The angle $\theta$ is in the x-y plane (in the orbital plane of $m_2$). The angle $\psi$ is the orbital plane of $m_1$. Note that as shown here in this diagram, the two planets are not actually passing conjunction.

**Problem 3.** Principal Lindblad Resonances

This problem is derived from Goldreich & Tremaine's (1980, ApJ, 241, 425, hereafter GT) paper on disk-satellite interactions.

In a coordinate system that attaches the origin to the (primary) star of mass $M$, the perturbation potential due to a (secondary) planet of mass $M_p$ reads

$$\phi^p(r,\theta,t) = -\frac{GM_p}{|\vec{r}-\vec{r}_p|} + \frac{GM_p}{|r_p|^3}\,\vec{r}_p\cdot\vec{r}$$

where $\vec{r}$ is the vector position (measured from the origin) where the potential is to be evaluated, and $\vec{r}_p$ is the vector displacement from the origin to the planet. Note that in equation (4) of GT, there is an error; their $(M_s/M_p)\Omega^2(r)$ should be replaced by $GM_s/r_s^3$. (This error does not propagate to the rest of their paper.)

It is useful to expand $\phi^p$ in a Fourier series:

$$\phi^p(r,\theta,t) = \sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty}\phi^p_{l,m}(r)\cos\{m\theta - [m\Omega_p + (l-m)\kappa_p]t\}$$

where $\Omega_p$ is the mean angular frequency of the planet (the rotational frequency of the guiding center of the planet's orbit), and $\kappa_p$ is the planet's epicyclic frequency (the frequency of radial oscillations due to non-zero eccentricity of the planet). In a frame that rotates at angular frequency $\Omega_p + (l-m)\kappa_p/m$, the perturbation potential is time-independent and has an $m$-fold azimuthal symmetry.

Assume that the planet's eccentricity is zero so that $|r_p|$ is a constant. Evaluate the strength of the "principal $m^{th}$ component" of the potential, $\phi^p_{m,m}(r)$. This expression is sufficient to describe the perturbation potential of a planet on a perfectly circular orbit, and it is the component that establishes "principal Lindblad resonances" (Galacto-speak) or "first-order mean-motion resonances" (planeto-speak) in the disk. Principal Lindblad resonances excited in a disk dominate the evolution of the semi-major axis of the planet; they are responsible for planet migration.

Express your answer in terms of Laplace coefficients (see the integral definition on page 237 of Murray and Dermott). Watch out for the cases $m=0$ and $m=1$. Compare your answer to equation (7) of GT.

**Problem 4.** Poincare, Lagrange, and Hamilton

Hamilton's equations read:

$$\dot{p}_i = \partial H/\partial q_i \tag{6}$$

$$\dot{q}_i = -\partial H/\partial p_i \tag{7}$$

Any set of variables $\{q_i, p_i\}$ that satisfy Hamilton's equations are called "canonical."

Unfortunately, the Keplerian osculating elements are not canonical variables. However, appropriately constructed combinations of the Kepler elements are canonical. One such combination is Poincare's set:

$$q_1 = \lambda \qquad p_1 = \sqrt{\mu a} \tag{8}$$

$$q_2 = -\tilde{\omega} \qquad p_2 = \sqrt{\mu a}\left(1 - \sqrt{1-e^2}\right) \tag{9}$$

$$q_3 = -\Omega \qquad p_3 = \sqrt{\mu a(1-e^2)}\,(1 - \cos i) \tag{10}$$

(See section 2.10 of MD but note that they add an extra $\mu^*$ into their equations for which we have no use.)

Insert Poincare's canonical variables as written above into Hamilton's equations to derive Lagrange's equations (6.145), (6.146), and (6.148)–(6.150). Ignore (6.147) for which we will have no use in this course. Everywhere you see $\epsilon$ in (6.145)–(6.150) replace it with $\lambda$ (see discussion on page 252).

This is more-or-less a plug-and-chug problem. Historically this is not the way Lagrange actually derived his equations. But the problem does highlight the fact that Lagrange's equations are really just Hamilton's equations, re-written in a nice practical way for celestial mechanicians.

Hint: $\partial R/\partial e = \sum_{i=1}^{3}(\partial R/\partial p_i)(\partial p_i/\partial e)$, and similarly for the other Kepler elements.