Astrophysical Fluid Dynamics – Problem Set 1

Readings: Shu pages 3–6; Tritton 5.1–5.5; White 4.2, 4.3, 4.5; Pringle & King Sections 1 through 1.4, and 1.7

Problem 1. Magic for Astronomers[1]

[10 points] Fill a tall glass with water, place a flat card covering the entire top of the glass, and upend the glass. Watch with amazement as the card remains suspended and water does not pour out!

Explain this phenomenon. What is pushing against the card to keep the water from falling? What is the tallest glass of water with which you can perform this trick?

[An alternative version of this trick, which you need not consider to receive full credit for this problem but may consider for fun, is to suck water up a thin straw, cap the top end of the straw with your finger, and behold the miracle of the water remaining suspended in the straw. This alternate trick is more complicated, however, because it is susceptible to effects due to the finite diameter of the straw (read: surface tension and the Rayleigh-Taylor instability, which you need not discuss, but may discuss if you would like)].

Problem 2. Tip of the Iceberg

[10 points] An iceberg is sighted with volume V above the water's surface. What volume of ice must reside below the water's surface? Take the density of berg ice, which is composed of frozen freshwater and full of bubbles, to be $\rho_{\rm ice} = 0.9\,{\rm g\,cm^{-3}}$, and the density of seawater to be $\rho_{\rm sea} = 1.03\,{\rm g\,cm^{-3}}$.

Try not to merely invoke Archimedes' principle, but instead try to derive the answer from hydrostatic equilibrium.

Problem 3. Scaling the Heights

[3 points for every part for a total of 18 points]

(a) Write down an expression for the variation of gas density ρ (units of mass per volume) with height z above the Earth's surface, assuming the gas is ideal, at constant temperature T, and in hydrostatic equilibrium. Take the gas to be made of a single kind of molecule of weight μ (units of mass).[2] Express in terms of the density at ground level

[1] Inspired by the most excellent Netflix series, "Magic for Humans".

[2] For this problem we take μ to have units of mass. In many other textbooks and papers, μ is dimensionless and gets multiplied by the mass of the hydrogen atom $m_{\rm H}$; in that case, μ is called the "mean molecular weight" (which always struck me as a terrible name, because weight has dimensions). We will go back and forth between the conventions; there should be no confusion.

$\rho_0 \equiv \rho(z=0)$ and the *density scale height* $h \equiv kT/(\mu g)$, where g is the gravitational acceleration. Neglect variations of g with z.

(b) Neglecting collisions with other molecules, what is the maximum height a molecule would attain if launched from $z = 0$ with a typical thermal velocity? Is this close to h?

This calculation is misleading insofar as it ignores collisions between molecules, which are crucial for describing air as a continuum fluid. Nevertheless, it provides a mnemonic for remembering the scale height, and it illustrates the sometimes surprisingly close connection between fluid mechanics and particle mechanics (kinetic theory).

(c) At what height z would you expect the formula in (a), which depends on the continuum hypothesis, to fail? This height marks the location of the *exobase* in planetary atmospheres. Just consider how intermolecular collisions validate the continuum approximation, and take σ to be the collisional cross-section.

(d) Write down an expression for the hydrostatic variation of gas density ρ with height z above the midplane of a circumstellar disk at radius r. As in (a), assume constant T and μ. Take the gravitational field to be that from the star alone (ignore the self-gravity of the disk). Work in the limit that $z \ll r$. Express in terms of the density at the midplane ρ_0 and the density scale height $h \equiv (kT/\mu)^{1/2}\Omega^{-1}$, where Ω is the Keplerian orbital angular frequency.

The height h is often written $h = c_s/\Omega$, where $c_s = (kT/\mu)^{1/2}$ is the speed of sound waves in gas that behaves isothermally [by behaving isothermally, we mean that the gas keeps the same temperature regardless of how it is compressed or expanded by the sound wave. This problem does not concern a sound wave *per se*, but people talk about the sound speed anyway because sound waves (pressure disturbances) are the means by which a medium establishes hydrostatic equilibrium.]

(e) As in (b), calculate the maximum height that a gas molecule would attain if launched upwards from the midplane with a typical thermal speed, ignoring collisions. Compare to h.

(f) For the disk to be geometrically "thin" ($h/r \ll 1$), what must be true about c_s/v_K? Here v_K is the Keplerian orbital velocity.

OPTIONAL Problem 4. To d/dt or Not to d/dt

[5 points] Give a physical explanation for why the left-hand side of the momentum equation reads $\rho du/dt$ and not $d(\rho u)/dt$, where u is velocity, ρ is density, and d/dt is the convective derivative. What is being assumed about the fluid parcel in the Lagrangian view?