Astrophysical Fluid Dynamics – Problem Set 10

Readings: Sturrock 14.1 (Course Reader) on MHD waves; Shu Chapter 22 on MHD waves; Blandford & Payne (1982) on magneto-centrifugal winds (Course Reader)

Problem 1. MHD Modes

The answers to this problem are all contained in Sturrock 14.1 (reprinted in the Course Reader) and Shu Chapter 22. You may use as much of Sturrock (or Shu) as you wish; I like deriving Sturrock’s results before using them, but however much you want to derive is up to you.

Consider a uniform, adiabatic, perfectly conducting medium of density $\rho_0$, pressure $P_0$, and adiabatic index $\gamma$, threaded with a magnetic field $\vec{B}_0 = B_0 \hat{z} = B_0 \hat{3}$. Perturb this medium with a small-amplitude (linear) wave of wavevector $\vec{k}$ and frequency $\omega$.

(a) [7 points] Write down the eigenmodes for Alfvén waves, restricting your attention to the case where the wavevector $\vec{k}$ is parallel to $\vec{B}_0$. That is, write down the Cartesian components for the perturbation velocity $\delta \vec{u} = [\delta u_1, \delta u_2, \delta u_3]$ and the perturbation magnetic field $\delta \vec{B} = [\delta B_1, \delta B_2, \delta B_3]$ when $k \parallel B_0$. Also write down the perturbation density $\delta \rho$ (a scalar) in terms of the magnitude of the perturbation velocity $\delta u \equiv |\delta \vec{u}|$.

Your solution should be complete up to an overall normalization constant, i.e., the wave amplitude. Express your answers for the eigenmode components in terms of the magnitude of the perturbation velocity $\delta u$. That is, all seven numbers $\delta u_1, \delta u_2, \delta u_3, \delta B_1, \delta B_2, \delta B_3, \delta \rho$ should be proportional to the free constant $\delta u$. Be careful of signs—these are important because they indicate phase relationships between the perturbations.

Express your answers in terms of the Alfvén velocity $u_A = \sqrt{B_0^2 / 4\pi \rho_0}$, the phase velocity $u_{ph} = \omega / |\vec{k}|$, and the sound speed $c_s \equiv \sqrt{\gamma P_0 / \rho_0}$.

Specify whether the mode is compressive or not, and whether it is transverse or longitudinal.

To gain physical intuition, you may find it helpful draw a picture of the mode, but this is not required.

(b) [7 points] Repeat (a) but for Alfvén waves with $\vec{k}$ perpendicular to $\vec{B}_0$. Do such waves propagate? Why or why not?

Note: Sturrock claims $\delta \vec{B} = 0$ for this case. But this is not necessarily what one would conclude from his equation 14.1.25, since $\cos \theta$ (\theta being the angle between $\vec{k}$ and $\vec{B}_0$) and the phase velocity (which he calls $v_\phi$ and we call $u_{ph}$) both go to zero as $\theta \to 90^\circ$. Full credit will be given whether you believe in Sturrock’s statement or whether, like me, you believe that $\delta \vec{B}$ should vary smoothly with $\theta$. The choice does not make any physical difference as you will see when you consider the mode’s propagation behavior.
(c) [7 points] Repeat (a) but for fast and slow magneto-sonic waves having \( \vec{k} \) parallel to \( \vec{B}_0 \). Does any wave not propagate?

(d) [7 points] Repeat (a) but for fast and slow magneto-sonic waves having \( \vec{k} \) perpendicular to \( \vec{B}_0 \). Does any wave not propagate?

**Problem 2. Magneto-Centrifugal Flinging**

[10 points] This problem is based on the classic paper on disk-driven outflows by Blandford & Payne (1982, reprinted in the Course Reader).

Consider a magnetized disk orbiting a central massive object. The central mass dominates the gravitational potential.

Examine a magnetic field line rooted to the disk midplane at distance \( a \) from the central object. Above the disk, the field line is straight but tilted away from the star at an angle \( \theta \) measured from the vertical (disk normal). The field line is mirror-symmetric about the disk plane (pointing toward the disk below the midplane and pointing away from the disk above the midplane). Take the field to be axisymmetric. Assume a given field line acts as a rigid wire, co-rotating with the disk where it is rooted (recall Ferraro’s law of iso-rotation).

For what angles \( \theta \) will disk plasma, starting at the disk midplane, slide unstably along a field line, away from the star, thereby producing a magneto-centrifugal outflow? Assume a fluid parcel can travel only parallel to the field (again recall Ferraro’s law). Assume that the pressure gradient \( \nabla P \) parallel to the field is negligible.

This looks like an MHD fluids problem, but with the given simplifications, it reduces to a bead-on-a-wire mechanics problem. Remember that there are two kinds of equilibrium: stable and unstable.