Astrophysical Fluid Dynamics – Problem Set 11

Readings: Course Reader selections from Binney & Tremaine on Toomre Q, and from Frank, King, & Raine on accretion disks

Problem 1. Toomre Q Mnemonics

A gas disk orbits an object of mass M. The disk is in vertical hydrostatic equilibrium with thickness h. You may wish to refer back to PS 1, Problem 3d where you solved for vertical hydrostatic equilibrium in a disk. You may use the results of that problem here, even though that earlier problem neglected disk self-gravity when calculating h; the error in using that h is on the order of unity for $Q \sim 1$ (and less serious for $Q \gg 1$).

(a) [4 points] Show that, to order-of-magnitude, $Q \equiv c_s\Omega/(\pi G\sigma) \sim 1$ at radius r corresponds to the condition $\rho \sim \rho_{\rm crit} \sim M/r^3$, where ρ is the gas density (mass per unit volume) at the disk midplane. Here c_s is the gas sound speed, G is the gravitational constant, σ is the mass per unit area of the disk, and Ω is the angular frequency, where all variables are evaluated locally at r.[1]

The condition you have derived is an easy way to remember the Toomre instability condition. The critical density $\rho_{\rm crit}$ is sometimes called the Roche density (and is just the mass of the central object divided into the sphere just enclosing your position).

(b) [3 points] Show that for $Q \sim 1$, the marginally unstable mode has a radial wavelength that scales as the gas vertical scale height h.

(c) [3 points] It is sometimes stated that $Q < 1$ corresponds to $M_{\rm disk}/M > h/r$. Explain why this statement makes rough sense.

Problem 2. Steady Alpha Disks

An accretion disk having surface density $\sigma(r)$ at radius r orbits a point mass M. The potential is Keplerian so the disk angular frequency $\Omega \propto r^{-3/2}$.

(a) [3 points] In class we had used the $\hat{\phi}$ component of the momentum equation to see that the radial accretion velocity u_r is given by:

$$\sigma u_r = \frac{\frac{\partial}{\partial r}\left(r^2 \int \tau_{r\phi} dz\right)}{r^2\left(\frac{\partial (r\Omega)}{\partial r} + \Omega\right)} . \tag{1}$$

[1] Technically the Toomre Q contains not the angular frequency Ω, but the radial epicyclic frequency κ. But the two frequencies are the same to within an order-unity factor; see the textbook by Binney & Tremaine, or the problem set question on radial epicycles and Rayleigh stability.

Model the stress in terms of a shear viscosity ν:

$$\tau_{r\phi} = \rho\nu r \frac{\partial \Omega}{\partial r} \tag{2}$$

and derive the mass accretion rate (the mass crossing a circle of radius r per unit time):

$$\dot{M} \equiv -2\pi\sigma u_r r = +6\pi \left[r\frac{\partial(\sigma\nu)}{\partial r} + \frac{\sigma\nu}{2} \right] \tag{3}$$

where $\sigma = \int \rho \, dz$. We have inserted a minus sign in the definition of $\dot{M}$ since astronomers like to say $\dot{M} > 0$ when mass flows radially in. If $\dot{M} < 0$, we have a "decretion" disk where mass flows radially outward. In general, $\dot{M}$ can vary with radius, causing mass to pile up or drain out locally in a non-steady evolution.

(b) [3 points] For the remaining parts of this problem, assume *steady* accretion: $\dot{M}$ is constant with radius (so mass does not pile up anywhere and $\partial\sigma/\partial t = 0$; you can see this last statement from the continuity equation we wrote down in class). From (1) derive:

$$\dot{M} = 3\pi\sigma\nu \, . \tag{4}$$

Hint: Solve for $y \equiv \sigma\nu$ in (3) and assume that $\sigma\nu =$ finite at $r = 0$.

(c) [3 points] Show that the result in (b) can be intuited to order-of-magnitude by taking the disk mass $M_{\rm disk} \sim \sigma r^2$ and the diffusion time $t \sim r^2/\nu$. Show also that the radial accretion speed $u_r \sim \nu/r$.

(d) [3 points] Accretion disks get hot. This is true whether we model them as viscous ("honey") or turbulent. For a gas parcel of mass Δm to go from a circular orbit at r_1 to a circular orbit at $r_2 < r_1$, it has to lose orbital energy (in our Kepler potential, the total orbital energy is $-GM\Delta m/(2r)$, so smaller r implies lower energy). That energy goes into heating the disk.

The heating rate per unit area is given very nearly by

$$D = \frac{3}{4\pi} \frac{GM\dot{M}}{r^3} \tag{5}$$

a result that can be derived in either the viscous picture (as is done in the Frank, King, & Raine book chapters reprinted in the Course Reader) or the turbulent picture (Balbus, Gammie, & Hawley 1994, MNRAS, 271). To order-of-magnitude, this result also makes sense: the energy lost by Δm in spiralling from r to $r/2$ is $\sim GM\Delta m/r$ (dropping all 2's); sending in Δm per time Δt means we are dissipating a power $\sim (GM\Delta m/r)/\Delta t \sim GM\dot{M}/r$; and spreading this power over an annulus of area $\sim r^2$ gives $\sim GM\dot{M}/r^3$, as in (5) above.

If the disk is able to radiate away this heat as a blackbody,[2] then we have

$$D = 2\sigma_{\rm SB} T^4 \tag{6}$$

or

$$\sigma_{\rm SB} T^4 = \frac{3}{8\pi} \frac{GM\dot{M}}{r^3} \tag{7}$$

where $\sigma_{\rm SB}$ is the Stefan-Boltzmann constant. The factor of 2 in (6) accounts for the top and bottom faces of the disk, both of which radiate into space.[3]

From (7), the assumption of a steady disk, and the assumption that the disk is vertically isothermal,[4] decide how the vertical disk scale height h scales with r. Sketch how the disk looks.

(e) [2 points] Now assume the kinematic viscosity $\nu = \alpha c_s h$, where c_s is the sound speed, and α is a dimensionless factor, assumed to be a strict constant. How does σ scale with r?

(f) [2 points] Suppose there is a pressure disturbance that perturbs the disk out of vertical hydrostatic equilibrium. Sound waves (pressure waves) try to even out the disturbance; the timescale to relax back to equilibrium is roughly the time it takes a sound wave to cross a scale height h. Call this the dynamical time t_z, and express in terms of Ω.

(g) [4 points] Find an approximate expression for how long it takes a temperature disturbance to equilibrate away. That is, consider perturbing the temperature so that $T \to T + \delta T$. The heat content of the disk has increased, but so has the blackbody flux to space. It takes some time for the perturbation flux to carry away the perturbation heat. Call this the cooling time $t_{\rm cool}$, and express in terms of Ω and α.

(h) [3 points] Find an approximate expression for how long it takes a mass disturbance—say a local accumulation of material on a radial length scale r—to diffuse radially away. Call this the viscous time $t_{\rm visc}$, and express in terms of Ω, α, and h/r.

Arrange t_z, $t_{\rm cool}$ and $t_{\rm visc}$ in increasing order, assuming $\alpha < 1$.

(i) [2 points] How does Toomre's Q scale with radius r? Accretion disks tend to be gravitationally unstable at large radii, leading some to surmise that the outer peripheries of quasar accretion disks

[2] It doesn't have to; it can radiate as a non-blackbody, or it might not radiate at all. If the latter, the accretion flow is called "advection-dominated"—the energy goes into heating the gas and just gets swept along radially. Advection-dominated accretion flows tend to be vertically thick/puffy.

[3] Side A and Side B of the vinyl record. (Do people still play vinyl?)

[4] A disk that is optically thick will not be vertically isothermal. It will be hotter at the midplane where most of the disk mass resides and where energy dissipation rates are highest. We are neglecting this effect.

/ protostellar accretion disks are fertile breeding grounds for starbursts / binary companion stars or brown dwarfs.[5]

Problem 3. Similarity Solutions for Accretion Disks

The governing equation for the surface density $\sigma(r,t)$ of a viscous accretion disk is a kind of diffusion equation:

$$\frac{\partial \sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\nu\sigma\right)\right] . \tag{8}$$

If the viscosity ν can be written as a power law in radius,

$$\nu \propto r^{\beta} , \tag{9}$$

then Lynden-Bell & Pringle (1974) showed that equation (8) admits a self-similar solution:

$$\sigma(r,t) = \frac{C}{3\pi\nu_1 R^{\beta}} T^{-(5/2-\beta)/(2-\beta)} \exp\left[-\frac{R^{(2-\beta)}}{T}\right] \tag{10}$$

where

$$R = r/r_1 \tag{11}$$
$$\nu_1 = \nu(r_1) \tag{12}$$
$$T = t/t_1 + 1 \tag{13}$$
$$t_1 = \frac{1}{3(2-\beta)^2}\frac{r_1^2}{\nu_1} \tag{14}$$

where r_1 is a fixed scaling radius (free parameter) and C is a normalization constant (also a free parameter). The quantity t_1 should ring a bell: it is the viscous diffusion time of the disc evaluated at r_1; see the previous Problem 2, and also Hartmann et al. (1998).

For the rest of this problem, assume $\beta = 0$ (spatially constant viscosity ν).

(a) [5 points] Sketch $\sigma(r)$ at three times, $t = (0, 10t_1, 100t_1)$, on one set of axes. Your plot need not be exact, but it should have the right scalings and orders of magnitude. Annotate your plot to answer the following questions:

1. There is a "break radius" $r_{\rm break}$ inside of which σ behaves as a near-perfect power law, and outside of which σ drops exponentially. Indicate the location of $r_{\rm break}$ on your plot and its relation to r_1. How does σ scale with r for $r \ll r_{\rm break}$? Just a proportionality is required.
2. How does $r_{\rm break}$ scale with time t for $t \gg t_1$? Just a proportionality is required.

[5] While wide stellar binaries are thought to form by gravitational instability, the story must be more complicated for compact stellar binaries (having separations less than 30 AU or so). Compact binaries are thought to start off wide, and then have their orbits shrunken—hardened is the technical term—by interacting with disk gas. In other words, the disk forces the binary companion to migrate inward.

3. At $r \ll r_{\rm break}$, how does σ scale with t for $t \gg t_1$? Just a proportionality is required.

(b) [2 points] How does the disc accretion rate $\dot{M}$ onto the central object (at $r = 0$) scale with time t for $t \gg t_1$? Just a proportionality is required.

Hint: At $r \ll r_{\rm break}$, the disc accretes quasi-steadily. That is, for times $t \ll t_{\rm break}$, where $t_{\rm break}$ is the viscous diffusion time at $r_{\rm break}$, the disc properties at $r \ll r_{\rm break}$ hardly change with time; the disc in this regime has viscously relaxed to a steady state. Under these circumstances you may use the results from Problem 2b above for steadily accreting discs.

(c) [3 points] Without referring to the similarity solution, write down a "zero-dimensional" order-of-magnitude model for a viscously diffusing accretion disc. The model is 0D because we are not spatially resolving the flow (we are not bothering to keep track of how flow properties change with cylindrical radius).

Say the disc at time t has a characteristic radius $r_{\rm disc}$, surface density σ, and viscosity ν. Further assume that the disc orbits a central mass $M_\star$ which dominates the mass in the system.

Use order-of-magnitude arguments to decide:

1. how the disk mass $M_{\rm disc}$ depends on σ and $r_{\rm disc}$
2. how the disk's total angular momentum L scales with σ, $r_{\rm disc}$, and t
3. how $r_{\rm disc}$ scales with t and ν
4. how σ scales with $r_{\rm disc}$
5. how σ scales with t
6. how the disc accretion rate $\dot{M}$ scales with t
7. how the disk mass $M_{\rm disc}$ scales with t

As a check, you may compare your order-of-magnitude results for how σ and $\dot{M}$ scale with t to the similarity solution (they should match). You may find the arguments in Problem 2c useful.

(d) OPTIONAL [5 points] Instead of a viscous disc, consider now a magneto-centrifugal wind-driven accretion disc. In a wind-driven disc, material accretes because a wind carries its angular momentum away to infinity (angular momentum goes out so mass goes in).

Continuity for a 2D axisymmetric cylindrical flow reads:

$$\frac{\partial \sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left(\sigma r u_r\right) \tag{15}$$

Assume $u_r = -u_0(r_0/r)$, where r_0 and $u_0 > 0$ are constants (so $u_r < 0$, which means disc material accretes inward). Derive an expression for $\sigma(r,t)$ in terms of u_0, r_0, an exponential decay time t_0, and an initial value $\sigma_0 = \sigma(r=0, t=0)$ at the origin. Compare your solution to the viscous similarity solution for $\beta = 0$ (which also gives $u_r \propto 1/r$) and comment.

Hint: given the assumptions, the equation is separable: $\sigma(r,t) = f(r)g(t)$ for separate functions f and g.