Problem 1. Magnetospheres

(a) [5 points] The solar wind blows across the Earth but is largely deflected away by the Earth’s magnetosphere. Estimate the radius $r_A$ of the Earth’s magnetosphere, in units of the Earth’s radius. This is the radius where the Earth’s magnetic pressure balances the pressure of the solar wind. Use whatever you need to from class, including previous problem sets. The surface field of the Earth is about 0.5 G (this is NOT the field strength at the magnetospheric boundary) and you may model the Earth’s field as a dipole.

In general, a magnetosphere marks the location where magnetic pressure (energy density) balances the pressure of some fluid flow. In the case of the Earth, the fluid is the solar wind. At 1 AU, the solar wind is supersonic, and we are well beyond the Sun’s magnetosphere, so we can reasonably ignore the wind’s magnetic and thermal energy densities, thus setting the energy density of Earth’s magnetic field equal to the bulk kinetic energy density of the wind:

$$\frac{1}{2} \rho_{sw} v_{sw}^2 = \frac{B(r_{ms})^2}{8\pi}.$$ 

From a dipole approximation, $B(r_{ms}) \sim B_\oplus (R_\oplus / r)^3$. Also, $\rho_{sw} \sim \mu_{sw} n_{sw}$, and we find

$$r_{ms} \sim \frac{B_\oplus^{1/3} R_\oplus}{v_{sw}^{1/3} (4\pi \mu_{sw} n_{sw})^{1/6}}.$$ 

Eugene’s values (see also PS 5):

$$\mu_{sw} \sim 2 \times 10^{-24} \text{ g}$$

$$n_{sw} \sim 10 \text{ cm}^{-3}$$

$$v_{sw} \sim 1000 \text{ km s}^{-1}$$

$$B_\oplus \sim 0.5 \text{ G}$$

These numbers yield $r_{ms} \approx 7R_\oplus$. To order of magnitude, $[r_{ms} \sim 10R_\oplus]$.

When the solar wind hits the magnetosphere, there is a collisionless shock in which the momentum of the solar wind is modified by magnetic forces (not by viscous forces, as would be the case in a collisional shock).
(b) [7 points] Consider now a different situation also involving a magnetosphere: an accretion disk orbiting a magnetized star (as discussed in lecture). The accretion disk is truncated at its inner edge by the star’s (closed) dipole field. Material from the disk diffuses (somehow, via instabilities of some kind) onto the magnetic field lines and is funneled onto the magnetic poles of the star. Our goal here is to derive a rough formula for the radius $r_A$ of the magnetosphere for this accretion disk case. This problem was drawn from Shapiro & Teukolsky’s excellent textbook, “Black Holes, White Dwarfs, and Neutron Stars”.

Figure 1 shows that the magnetospheric boundary is actually a fuzzy one, having some radial thickness $\delta$ over which flow variables like density, velocity, and magnetic field change radially. Figure 1 also shows that the disk distorts the stellar magnetic field over a vertical length scale of order the disk vertical thickness, $H$ — specifically, the disk creates toroidal field ($B_\phi$) from a purely poloidal (and mostly vertical $B_z$) stellar field, as a consequence of flux freezing; the rotating disk tries to “pull” field lines into the azimuthal direction. For disks, $H \ll r$, where $r$ is the disk radius.

Assuming steady-state and axisymmetry, use the azimuthal $\hat{\phi}$-component of the momentum equation to derive the following order-of-magnitude relation:

$$\frac{\dot{M} v_\phi}{r \delta} \sim B_z^2$$  \hspace{1cm} (1)
valid near the disk midplane at the magnetospheric boundary. Here \( v_\phi \) is the disk gas velocity in the azimuthal direction, \( \dot{M} \sim 2\pi \rho H v_r r \) is the disk accretion rate (mass per time crossing a circle of radius \( r \); if this statement is not clear to you, try showing it to yourself), \( \rho \) is the gas density, and \( v_r \) is the gas radial velocity.

It is not obvious what \( \delta \) should be, but there are two end-member cases we can consider: either \( \delta \sim r \) (the largest relevant length scale) or \( \delta \sim H \) (the smallest relevant length scale; Rayleigh stability supports the statement \( \min \delta \sim H \), as you will see in Problem 2 of this set).

To derive (1), assume \( |B_r|/\delta \ll |B_z|/H \) and \( |B_\phi| \sim |B_z| \) near the magnetospheric boundary. Both assumptions should appear plausible from studying Figure 1.

The \( \hat{\phi} \) component of the momentum equation reads:

\[
\rho \frac{v_r}{r} \frac{\partial}{\partial r} (rv_\phi) = \frac{1}{4\pi} \left[ (\nabla \times \vec{B}) \times \vec{B} \right]_\phi
\]

where we have used our trusty Course Reader to write down the \( \phi \)-component of the advective term on the LHS, and where we have used \( \partial / \partial t = 0 \) (steady) and \( (\nabla P)_\phi = (\nabla \Phi)_\phi = 0 \) (axisymmetry of scalar fields). The only term left on the RHS is the Lorentz force. To get \( \hat{\phi} \) from a cross product, we cross \( \hat{r} \) into \( \hat{z} \) or \( \hat{z} \) into \( \hat{r} \):

\[
\rho \frac{v_r}{r} \frac{\partial}{\partial r} (rv_\phi) = -\frac{1}{4\pi} \left[ (\nabla \times \vec{B}) \right]_r B_z + \frac{1}{4\pi} \left[ (\nabla \times \vec{B}) \right]_z B_r
\]

Again from the Course Reader,

\[
\rho \frac{v_r}{r} \frac{\partial}{\partial r} (rv_\phi) = \frac{1}{4\pi} \frac{\partial B_\phi}{\partial z} B_z + \frac{1}{4\pi} \frac{1}{r} \frac{\partial (r B_\phi)}{\partial r} B_r
\]

Now write down the order-of-magnitude version of this equation, replacing \( \partial r \sim \delta \) and \( \partial z \sim H \):

\[
\frac{\rho v_r v_\phi}{\delta} \sim \frac{1}{4\pi} \frac{B_\phi B_z}{H} + \frac{1}{4\pi} \frac{B_\phi B_r}{\delta}
\]

We are told to assume that \( |B_r|/\delta \ll |B_z|/H \), so we drop the second term on the RHS. The assumption makes sense because \( \delta > H \) (as stated in the problem) and also because \( |B_r| \ll |B_z| \) close to the disk midplane (field lines there are primarily vertical). So we have:

\[
\frac{\rho v_r v_\phi}{\delta} \sim \frac{1}{4\pi} \frac{B_\phi B_z}{H}
\]

Now again do as the problem says and replace \( B_\phi \) with \( B_z \), which looking at Figure 1 is reasonable. Finally throw the \( 4\pi H \) over to the LHS, and recognize \( \dot{M} \sim 2\pi \rho H v_r r \) to find the desired relation:

\[
\frac{\dot{M} v_\phi}{r \delta} \sim B_z^2
\]

where we discarded the 2 since this is an order-of-magnitude relation.

Suppose \( \delta \sim r \): solve (1) for \( r = r_A \) as a function of \( B_\star \) (the stellar surface field), \( R_\star \) (the stellar radius), \( M_\star \) (the stellar mass), \( \dot{M} \), and fundamental constants. Assume a stellar dipole field. Compare your expression to the one we derived in class for a spherically accreting stellar dipole.
Replace \(v_\phi = \sqrt{GM_*/r}\) and \(B_z \sim B_*(R_*/r)^3\), and solve for

\[
r \sim r_A \sim \frac{B_*^{4/7} R_*^{12/7}}{M^{2/7}(GM_*)^{1/7}}
\]

which we recognize as pretty much identical to the expression we got in class for a spherically accreting stellar dipole. This expression is what is used most commonly in the literature.

Now suppose \(\delta \sim H\): solve for \(r = r_A\) as above, now including \(H\) as part of your answer.

Straightforward algebra gives

\[
r \sim r_A \sim \frac{B_*^{4/9} R_*^{4/3} H^{2/9}}{M^{2/9}(GM_*)^{1/9}}
\]

(c) [3 points] Estimate the radius \(r_A\) of the magnetosphere of a young star accreting from a disk, assuming \(\delta \sim r\).

Use parameters typical of a T Tauri star: \(M_* \sim 1M_\odot\), \(R_* \sim 3R_\odot\), a dipole field of surface strength \(B_* \sim 1\) kG, and the median measured accretion rate of \(\dot{M} \sim 10^{-8}M_\odot\) yr\(^{-1}\). Express your answer for \(r_A\) in units of AU.

It is known that extrasolar sub-Neptune planets appear less frequently at distances < 0.1 AU from their central stars. This may be because their parent disks were truncated at \(\sim 0.1\) AU by their host star magnetospheres (no disk, no planet formation). Compare your estimate for \(r_A\) to this observed drop-off radius in planet occurrence (see Lee & Chiang 2017).

We showed above in part (b) that if \(\delta \sim r\) then the situation for disk accretion is the same as the situation for spherical accretion. In this case, the magnetosphere is located at

\[
r_A \sim \frac{R_*^{12/7} B_*^{4/7}}{M^{2/7}(GM_*)^{1/7}}.
\]

Inserting the given values \(M_*, R_*, B_*\) and \(\dot{M}\), we find \(r_A \sim 0.1\) AU, which is encouragingly close to the observed planet cut-off radius.

**Problem 2. Radial Epicycles and Rayleigh Stability**

Consider an axisymmetric gas disk in 2D. The radial and azimuthal components of the momentum equation read:

\[
\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} - \frac{u_\phi^2}{r} = \frac{\partial \Phi}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial r}
\]

\[
\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi u_r}{r} = 0
\]
where \( u_r \) is the radial velocity, \( u_\phi \) is the azimuthal velocity, \( r \) is the disk radius, \( P \) is pressure, \( \rho \) is density, and \( \Phi \) is the gravitational potential (which may include a self-gravitational component but doesn’t have to). On the left-hand side we have expanded out all the terms in \((\vec{\mathbf{u}} \cdot \nabla)\vec{\mathbf{u}}\).

The convective derivative of any scalar \( x \) (the derivative following some scalar\(^1\) property \( x \) of a fluid parcel) reads:

\[
\frac{dx}{dt} = \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + u_\phi \frac{1}{r} \frac{\partial}{\partial \phi} \right) x = \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} \right) x
\]

(12)

where the last equality follows because the disk is assumed axisymmetric at all times. Thus we are really tracking how \( x \) changes with the radial motion of entire (circular) rings of gas.

(a) [3 points] Prove from the \( \phi \)-momentum equation that

\[
\frac{d}{dt}(ru_\phi) = 0
\]

(13)

The quantity \( ru_\phi \) is the specific angular momentum of gas, which you have just shown is conserved (even as the rings of gas move radially). Call \( \ell_z = ru_\phi = \text{constant} \).

Multiply (11) by \( r \) to get

\[
r \frac{\partial u_\phi}{\partial t} + ru_r \frac{\partial u_\phi}{\partial r} + u_\phi u_r = 0
\]

(14)

which can be re-written as

\[
\frac{\partial ru_\phi}{\partial t} + u_r \frac{\partial ru_\phi}{\partial r} = 0
\]

(15)

which proves the desired result.

(b) [5 points] For the remaining parts of this problem, consider only a single “test ring” (read: test particle) having a strictly constant \( \ell_z = ru_\phi \).

Show that the \( r \)-momentum equation for this single test ring can be written as

\[
\frac{du_r}{dt} = -\frac{\partial \Phi_{\text{eff}}}{\partial r}
\]

(16)

where the effective potential

\[
\Phi_{\text{eff}} \equiv \frac{\ell_z^2}{2r^2} + \Phi + h
\]

(17)

where we emphasize that \( \ell_z \) is a strict constant, and \( h = \int (1/\rho) dP \) is the enthalpy (which should be familiar from the Bernoulli constant). The effective potential contains a centrifugal component (sometimes called the “centrifugal barrier”), the gravitational component, and a pressure component.

\(^1\)I emphasize scalar here because you can also take the convective derivative of a vector, which of course is what is done for, e.g., the inertial term \((\vec{\mathbf{u}} \cdot \nabla)\vec{\mathbf{u}}\). The terms \(-u_\phi^2/r\) and \(u_\phi u_r/r\) in (10) and (11) arise from taking the convective derivative of the unit vectors \(\hat{r}\) and \(\hat{\phi}\)—these are non-zero, even for axisymmetric flows.
Re-write (10) as
\[
\frac{du_r}{dt} = \frac{u_0^2}{r} - \frac{\partial \Phi}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial r}
\]  
(18)

Replace \(u_\phi\) with \(\ell_z/r\):
\[
\frac{du_r}{dt} = \frac{\ell_z^2}{r^3} - \frac{\partial \Phi}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial r}
\]  
(19)
\[
= - \frac{\partial}{\partial r} \left( \frac{\ell_z^2}{2r^2} \right) - \frac{\partial \Phi}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial r}
\]  
(20)
\[
= - \frac{\partial}{\partial r} \Phi_{\text{eff}}
\]  
(21)
as desired.

(c) [5 points] The equilibrium position \(r_0\) of the ring of gas is given by \(du_r/dt = 0\). For small-amplitude radial displacements about \(r_0\), we have
\[
\frac{du_r}{dt} \overset{\ddot{r}}{=} - \left. \frac{\partial \Phi_{\text{eff}}}{\partial r} \right|_{r_0} - \left. \frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} \right|_{r_0} (r - r_0)
\]  
(22)
\[
= - \left. \frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} \right|_{r_0} (r - r_0)
\]  
(23)
which we recognize as the equation for a linear spring with frequency \(\kappa\) given by
\[
\kappa^2 = \frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2}
\]  
(24)
where henceforth it is understood that we are evaluating quantities at \(r_0\) and so drop the \(\big|_{r_0}\) notation. In particular we understand that \(\ell_z\) in \(\Phi_{\text{eff}}\) is specific to \(r_0\) so that \(\ell_z = u_\phi r_0 = \text{constant}\) (the specific angular momentum of the ring is conserved while it undergoes radial oscillations).

Neglect for now the enthalpy term in \(\Phi_{\text{eff}}\) (in disks, \(h\) is generally small compared to \(\Phi\); for now we drop it entirely). Show that
\[
\kappa^2 = 4\Omega^2 + r \frac{\partial (\Omega^2)}{\partial r}
\]  
(25)
where \(\Omega = u_\phi / r\) is the angular frequency. Show also that
\[
\kappa^2 = \frac{1}{r^3} \frac{\partial (\Omega r^2)^2}{\partial r}
\]  
(26)
Thus when \(\kappa^2 > 0\)—corresponding to flows that are “Rayleigh-stable”—the specific angular momentum \(\Omega r^2\) of the background disk increases with increasing radius.

To derive the above relations, recognize from (10) that the equilibrium rotation profile (for zero-pressure disks) is just given by the gravitational potential via
\[
\Omega^2 r = \frac{u_0^2}{r} = \frac{\partial \Phi}{\partial r}
\]  
(27)
Note this last equation uses $\Phi$, not $\Phi_{\text{eff}}$.

If $\Phi_{\text{eff}} = \frac{\ell^2}{2r^2} + \Phi$, then

$$\frac{\partial \Phi_{\text{eff}}}{\partial r} = -\frac{\ell^2}{r^3} + \frac{\partial \Phi}{\partial r}$$

(28)

and

$$\frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} = +\frac{3\ell^2}{r^4} + \frac{\partial^2 \Phi}{\partial r^2}.$$  

(29)

Differentiate (27) to find

$$\frac{\partial^2 \Phi}{\partial r^2} = \frac{\partial}{\partial r} (\Omega^2 r)$$

(30)

and insert into (29):

$$\frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} = +\frac{3\ell^2}{r^4} + \frac{\partial}{\partial r} (\Omega^2 r)$$

(31)

$$= 3\Omega^2 + r \frac{\partial \Omega^2}{\partial r} + \Omega^2$$

(32)

$$= 4\Omega^2 + r \frac{\partial \Omega^2}{\partial r}$$

(33)

as desired. Equation (26) is easily verified.

(d) [2 points] Astronomers typically assume that $\kappa \sim \Omega$. What is $\kappa$ for a galactic disk with a flat rotation profile $u_\phi = \Omega r = \text{constant}$, in units of $\Omega$? What is $\kappa$ for a point-mass (a.k.a. Kepler) potential?

For $\Omega \propto r^{-1}$, $\boxed{\kappa = \sqrt{2}\Omega}$. For $\Omega \propto r^{-3/2}$, $\boxed{\kappa = \Omega}$.

(e) [10 points] Now restore the enthalpy $h$ to consider pressure gradients. Take the gas to have sound speed $c_s$ and recall from PS 1, Problem 3f that the disk’s vertical scale height $H$ is given by $H/r = c_s/(\Omega r)$. A disk has $H/r < 1$ (otherwise it wouldn’t be called a disk) which implies the enthalpy $h < \Phi$ (you can check this to order-of-magnitude).

Consider the inner edge of a disk, where the gas pressure decreases (going inward) over a radial length scale $\delta$. Assume a point-mass gravitational potential. Make an order-of-magnitude estimate for the smallest value $\delta$ can be before gas at the disk edge becomes Rayleigh-unstable ($\kappa^2 < 0$). Rayleigh stability sets a limit to how sharp disk edges can be (a fact of possible relevance to determining the structure of gaps opened by planets in circumstellar disks).

Hint: the pressure profile $P(r)$ of a smooth annular disk must have a negative second derivative somewhere.

A rotating object meets the criterion for Rayleigh stability if its epicyclic frequency, $\kappa$, is real.

$$\kappa^2 = \frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} > 0,$$
where
\[
\Phi_{\text{eff}} = \frac{l_z^2}{2r^2} + \Phi + h.
\]
Specific angular momentum \(\ell_z = r^2\Omega\) is conserved, so
\[
\frac{\partial^2}{\partial r^2} \left( \frac{l_z^2}{2r^2} \right) = \frac{1}{2} l_z^2 \left( \frac{6}{r^4} \right) = 3\Omega^2.
\]
For a disk orbiting a star of mass \(M\), with negligible disk self-gravity,
\[
\Phi = -\frac{GM}{r}, \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial r^2} = -\frac{2GM}{r^3}.
\]
In the case of Keplerian rotation, \(\frac{-2GM}{r^3} = -2\Omega^2\).

Now we consider:
\[
\frac{\partial^2 h}{\partial r^2} = \frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r} \left( \frac{1}{\rho} \int \frac{1}{\rho} dP \right) \right]
\]
\[
= \frac{\partial}{\partial r} \left( \frac{\partial P}{\partial r} \frac{\partial}{\partial P} \int \frac{1}{\rho} dP \right)
\]
\[
= \frac{\partial}{\partial r} \left( \frac{\partial P}{\partial r} \rho^{-1} \right)
\]
\[
= -\rho^{-2} \frac{\partial \rho}{\partial r} \frac{\partial P}{\partial r} + \rho^{-1} \frac{\partial^2 P}{\partial r^2}
\]

At the inner edge of the disk, we imagine both \(\rho\) and \(P\) increases with increasing \(r\) so that the first term on the RHS is negative. What about the sign of the second term? This second derivative will also be negative—as \(r\) increases, eventually \(P\) will level off, in the main disk away from the inner edge, and so the slope \(\partial P/\partial r\) must be decreasing with increasing \(r\). One could counter-argue by imagining a region where \(\partial^2 P/\partial r^2\) is positive—a tail of gas that trickles to nothing as one approaches the central object—but this concave-up region must eventually give way to a concave-down region, because the main body of the disk—think of it as an annulus—is concave-down. It is this concave-down segment of the inner edge whose length scale \(\delta\) we are trying to bound.

All of this is to argue that we can write—with minus signs!—to order-of-magnitude:
\[
\frac{\partial^2 h}{\partial r^2} \sim -\frac{P}{\rho \delta^2} - \frac{P}{\rho \delta^2} \sim -\frac{c_s^2}{\delta^2}
\]

Putting it all together, we have
\[
\kappa^2 \sim 3\Omega^2 - 2\Omega^2 - \frac{c_s^2}{\delta^2} > 0.
\]
This yields \(\delta > c_s/\Omega\), indicating that \(\delta\) must be larger than the scale height of the disk to maintain Rayleigh stability.