Problem 1. Radial Epicycles

Consider an axisymmetric gas disk in 2D. The radial and azimuthal components of the momentum equation read:

\(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} - \frac{u_r^2}{r} = -\frac{\partial \Phi}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial r} \) \hspace{1cm} (1)

\(\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi u_r}{r} = 0 \) \hspace{1cm} (2)

where \(u_r\) is the radial velocity, \(u_\phi\) is the azimuthal velocity, \(r\) is the disk radius, \(P\) is pressure, \(\rho\) is density, and \(\Phi\) is the gravitational potential (which may include a self-gravitational component but doesn’t have to). On the left-hand side we have expanded out all the terms in \((\vec{u} \cdot \nabla)\vec{u}\).

The convective derivative of any scalar \(x\) (the derivative following some scalar\(^1\) property \(x\) of a fluid parcel) reads:

\(\frac{dx}{dt} = \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\phi}{r} \frac{\partial}{\partial \phi} \right) x = \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} \right) x \) \hspace{1cm} (3)

where the last equality follows because the disk is assumed axisymmetric at all times. Thus we are really tracking how \(x\) changes with the radial motion of entire (circular) rings of gas.

(a) Prove from the \(\phi\)-momentum equation that

\(\frac{d}{dt}(ru_\phi) = 0 \) \hspace{1cm} (4)

The quantity \(ru_\phi\) is the specific angular momentum of gas, which you have just shown is conserved (even as the rings of gas move radially). Call \(\ell_z = ru_\phi = \text{constant}\).

Multiply (2) by \(r\) to get

\(\frac{r \partial u_\phi}{\partial t} + ru_r \frac{\partial u_\phi}{\partial r} + u_\phi u_r = 0 \) \hspace{1cm} (5)

which can be re-written as

\(\frac{\partial (ru_\phi)}{\partial t} + u_r \frac{\partial (ru_\phi)}{\partial r} = 0 \) \hspace{1cm} (6)

which proves the desired result.

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\(^1\) I emphasize scalar here because you can also take the convective derivative of a vector, which of course is what is done for, e.g., the inertial term \((\vec{u} \cdot \nabla)\vec{u}\). The terms \(-u_r^2/r\) and \(u_\phi u_r/r\) in (1) and (2) arise from taking the convective derivative of the unit vectors \(\hat{r}\) and \(\hat{\phi}\)—these are non-zero, even for axisymmetric flows.
(b) Show that the r-momentum equation can be written as
\[
\frac{du_r}{dt} = -\frac{\partial \Phi_{\text{eff}}}{\partial r}
\] (7)
where the effective potential
\[
\Phi_{\text{eff}} \equiv \frac{\ell_z^2}{2r^2} + \Phi + h
\] (8)
where \( h = \int (1/\rho) dP \) is the enthalpy (which should be familiar from the Bernoulli constant). The effective potential contains a centrifugal component (sometimes called the “centrifugal barrier”), the gravitational component, and a pressure component.

Re-write (1) as
\[
\frac{du_r}{dt} = -\frac{u_\phi^2}{r} \frac{\partial \Phi}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial r}
\] (9)
Replace \( u_\phi \) with \( \ell_z/r \):
\[
\frac{du_r}{dt} = +\frac{\ell_z^2}{r^3} - \frac{\partial \Phi}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial r}
\] (10)
\[
= -\frac{\partial}{\partial r} \left( \frac{\ell_z^2}{2r^2} \right) - \frac{\partial \Phi}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial r}
\] (11)
\[
= -\frac{\partial}{\partial r} \Phi_{\text{eff}}
\] (12)
as desired.

(c) The equilibrium position \( r_0 \) of a ring of gas is given by \( \frac{du_r}{dt} = 0 \). For small-amplitude radial displacements about \( r_0 \), we have
\[
\frac{du_r}{dt} = \ddot{r} = -\frac{\partial \Phi_{\text{eff}}}{\partial r} \bigg|_{r_0} - \frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} \bigg|_{r_0} (r - r_0)
\] (13)
\[
= -\frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} \bigg|_{r_0} (r - r_0)
\] (14)
which we recognize as the equation for a linear spring with frequency \( \kappa \) given by
\[
\kappa^2 = \frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2}
\] (15)
where henceforth it is understood that we are evaluating quantities at \( r_0 \) and so drop the \( |_{r_0} \) notation. In particular we understand that \( \ell_z \) in \( \Phi_{\text{eff}} \) is specific to \( r_0 \) so that \( \ell_z = u_\phi r_0 = \) constant (the specific angular momentum of the ring is conserved while it undergoes radial oscillations).

Neglect for now the enthalpy term in \( \Phi_{\text{eff}} \) (in disks, \( h \) is generally small compared to \( \Phi \); for now we drop it entirely). Show that
\[
\kappa^2 = 4\Omega^2 + r \frac{\partial (\Omega^2)}{\partial r}
\] (16)
where \( \Omega = u_\phi / r \) is the angular frequency. Show also that

\[
\kappa^2 = \frac{1}{r^3} \frac{\partial (\Omega r^2)^2}{\partial r}
\]

(17)

Thus when \( \kappa^2 > 0 \)—corresponding to flows that are “Rayleigh-stable”—the specific angular momentum \( \Omega r^2 \) of the background disk increases with increasing radius.

To derive the above relations, recognize from (1) that the equilibrium rotation profile (for zero-pressure disks) is just given by the gravitational potential via

\[
\Omega^2 r = \frac{u_\phi^2}{r} = \frac{\partial \Phi}{\partial r}
\]

(18)

Note this last equation uses \( \Phi \), not \( \Phi_{\text{eff}} \).

If \( \Phi_{\text{eff}} = \ell^2 z / 2r^2 + \Phi \), then

\[
\frac{\partial \Phi_{\text{eff}}}{\partial r} = -\frac{\ell^2}{r^3} + \frac{\partial \Phi}{\partial r}
\]

(19)

and

\[
\frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} = \frac{3\ell^2}{r^4} + \frac{\partial^2 \Phi}{\partial r^2}
\]

(20)

Differentiate (18) to find

\[
\frac{\partial^2 \Phi}{\partial r^2} = \frac{\partial}{\partial r} (\Omega^2 r)
\]

(21)

and insert into (20):

\[
\frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} = \frac{3\ell^2}{r^4} + \frac{\partial}{\partial r} (\Omega^2 r)
\]

(22)

\[
= 3\Omega^2 + r \frac{\partial \Omega^2}{\partial r} + \Omega^2
\]

(23)

\[
= 4\Omega^2 + r \frac{\partial \Omega^2}{\partial r}
\]

(24)

as desired. Equation (17) is easily verified.

(d) Astronomers typically assume that \( \kappa \sim \Omega \). What is \( \kappa \) for a galactic disk with a flat rotation profile \( u_\phi = \Omega r \) constant, in units of \( \Omega \)? What is \( \kappa \) for a point-mass (a.k.a. Kepler) potential?

For \( \Omega \propto r^{-1} \), \( \kappa = \sqrt{2} \Omega \). For \( \Omega \propto r^{-3/2} \), \( \kappa = \Omega \).

(e) Now restore the enthalpy \( h \) to consider pressure gradients. Take the gas to have sound speed \( c_s \) and recall from PS 1, Problem 3f that the disk’s vertical scale height \( H \) is given by \( H/r = c_s / (\Omega r) \). A disk has \( H/r < 1 \) (otherwise it wouldn’t be called a disk!) which implies the enthalpy \( h < \Phi \) (you can check this to order-of-magnitude).

Consider the inner edge of a disk, where the gas pressure decreases (going inward) over a radial length scale \( \delta \). Assume a point-mass gravitational potential. Make an order-of-magnitude estimate
for the smallest value $\delta$ can be before gas at the disk edge becomes Rayleigh-unstable ($\kappa^2 < 0$). Rayleigh stability sets a limit to how sharp disk edges can be (a fact of possible relevance to determining the structure of gaps opened by planets in circumstellar disks).

A rotating object meets the criterion for Rayleigh stability if its epicyclic frequency, $\kappa$, is real.

$$\kappa^2 = \frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} > 0,$$

where

$$\Phi_{\text{eff}} = \frac{l_z^2}{2r^2} + \Phi + h.$$

Specific angular momentum $l_z = r^2 \Omega$ is conserved, so

$$\frac{\partial^2}{\partial r^2} \left( \frac{l_z^2}{2r^2} \right) = \frac{1}{2} l_z^2 \left( \frac{6}{r^4} \right) = 3\Omega^2.$$

For a disk orbiting a star of mass $M$, with negligible disk self-gravity,

$$\Phi = -\frac{GM}{r}, \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial r^2} = -\frac{2GM}{r^3}.$$

In the case of Keplerian rotation, $-\frac{2GM}{r^3} = -2\Omega^2$.

At the inner edge of the disk, $P$ and its associated entropy term $h \equiv \int \frac{dP}{\rho}$ drop over a distance $\delta$, so we can estimate

$$\frac{\partial^2 h}{\partial r^2} \sim -\frac{h}{\delta^2}.$$

In a more rigorous manipulation of $\frac{\partial^2 h}{\partial r^2}$, we would use $dP = c_s^2 d\rho$ and obtain the negative sign above from a $\frac{\partial}{\partial r} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right)$ term.

Furthermore, we approximate that

$$h \sim \frac{P}{\rho} = \frac{c_s^2}{\gamma} \sim c_s^2.$$

We therefore have

$$\kappa^2 \sim 3\Omega^2 - 2\Omega^2 - \frac{c_s^2}{\delta^2} > 0.$$

This yields $\boxed{\delta > c_s/\Omega}$, indicating that $\delta$ must be larger than the scale height of the disk to maintain Rayleigh stability.

**Problem 2. Spiral Density Waves**

Consider spiral disturbances of the form $\exp(i(k_r r + m\phi - \omega t))$ in a disk. Note that $k_r < 0$ for leading waves and $k_r > 0$ for trailing waves.
This problem explores basic kinematics of waves in disks. By solving part (c), you will see why the dispersion relation that we presented in class cannot be used to study the stability of non-axisymmetric \((m \neq 0)\) waves, because that dispersion relation assumes that waves always remain tightly wound and that \(k\) is constant.

(a) Show that \(\cot i = |k_r r/m|\), where \(i\) is the pitch angle between the tangent to a spiral arm and the local circle \(r = \text{constant}\).

This formula is valid for any value of \(i\). In the special case that waves are tightly wound (permitting a local analysis), \(i \ll 1\) and \(|kr| \gg 1\).

A spiral arm is the locus of constant phase \((k_r r - \omega t + m\phi = \text{constant})\). If we consider the spiral disturbance at a fixed time and look at two neighboring points given by \((r, \phi)\) and \((r + \Delta r, \phi + \Delta \phi)\), we know that if they are on the same spiral arm

\[
k_r r + m\phi = \text{constant} = k_r (r + \Delta r) + m(\phi + \Delta \phi).
\]

Therefore

\[
k_r \Delta r + m \Delta \phi = 0.
\]

Then

\[
\frac{k_r r}{m} = -\frac{r \Delta \phi}{\Delta r}.
\]

We construct \(\cot(i)\) from Figure 1, valid for arbitrary \(i\) (but small \(\Delta \phi\)). As it is shown in Figure 1, \(\cot(i) = (r \Delta \phi)/(\Delta r)\). Combining this result with equation (27) we get

\[
\cot(i) = \frac{|k_r r|}{m}.
\]

(b) Consider a parcel of gas on a purely circular orbit with angular frequency \(\Omega\). With what frequency do the spiral arms wash over this parcel of gas, for \(m \neq 0\)? Express this frequency in terms of the variables given.

This frequency is sometimes called the Doppler-shifted forcing frequency in the disk literature (“Doppler-shifted” to remind you that everything is in relative circular motion, and “forcing” to remind you that a given gas parcel is being kicked by the spiral disturbance at this frequency.) We gave the answer in class without derivation; here you are asked to give a derivation (but not the full derivation of the dispersion relation—the problem asks only that you derive the forcing frequency in terms of the variables given in the problem).

Hint: The wave has \(m\) arms, and \(\omega\) is the frequency at which a point fixed in inertial space gets kicked by the wave.

Examine the defining equation for the disturbance, \(\exp i(k_r r - \omega t + m\phi)\). The parcel of gas which is moving on a purely circular orbit has \(\phi = \phi_0 + \Omega t\). Plug this into the defining equation for the disturbance to find that the disturbance behaves as \(\exp i k_r r \exp -i(\omega - m\Omega)t\) (times an
Figure 1: Pitch angle, valid for arbitrarily $i$ (by construction, $\Delta \phi \ll 1$)
uninteresting offset phase that depends on \( \phi_0 \)). So the disturbances wash over this parcel with frequency \( \omega - m\Omega \).

Often dynamicists define a “pattern speed” \( \Omega_p \) in lieu of \( \omega \), such that \( 2\pi/\Omega_p \) represents the time it takes a given spiral arm (e.g., spiral arm #2 out of a total of 5) to rotate a full \( 2\pi \). Then \( \Omega_p = \omega/m \), and the Doppler-shifted forcing frequency \( \omega - m\Omega = m(\Omega_p - \Omega) \).

(c) At the so-called “corotation circle,” the parcel does not get kicked by the wave, because it is moving at just the right angular frequency \( \Omega \) to keep up with the wave (the parcel is “corotating” with the wave). Every wave has its own corotation circle.

Consider a non-axisymmetric wave \( (m \neq 0) \) at whose corotation circle \( r = r_0 \) and \( \Omega = \Omega_0 \). Go into the frame rotating at \( \Omega_0 \). The wave is generally NOT stationary in this frame. It would be stationary if the entire disk were in solid body rotation. But because disks are generally differentially rotating \( (\Omega \text{ varies with } r) \), the wavefronts get turned by the background shear. Leading waves become trailing waves at the corotation circle.

Derive expressions for \( k_r(t) = 2\pi/\lambda_r(t) \) (the radial wavenumber) and \( k_\phi(t) = 2\pi/\lambda_\phi(t) \) (the wavenumber in the azimuthal direction) in terms of their initial values \( k_r(0) \) and \( k_\phi(0) \), \( m \), \( r_0 \), \( \Omega_0 \), and \( t \). Restrict the analysis to Kepler shear \( (\Omega \propto r^{-3/2}) \) and to a narrow annulus around corotation \( (\Delta r \ll r) \). The latter restriction constitutes the so-called “shearing sheet” approximation. In the shearing sheet, the flow appears Cartesian, and the wavefronts remain locally straight while they turn from leading to trailing.

You may find the following Figure 2, showing a snapshot of a leading wave near corotation, helpful.

![Figure 2: Wavefronts in the shearing sheet. The arrows on the right-hand side show the fluid velocities associated with the background shear (assumed for this problem to be Keplerian).](image)

Refer to Figure 3, which is an expanded version of Figure 2. First recognize that the azimuthal wavelength, \( \lambda_\phi \), does not change from the shear. So \( k_\phi = k_\phi(0) \).

Next see that \( \lambda_r = \lambda_\phi / \tan \gamma \), where \( \gamma \) is the angle that the wavefront makes w.r.t. the radial.
So we have $k_r = k_\phi \tan \gamma$. We'll adopt the sensible convention that $\gamma < 0$ gives $k_r < 0$ (trailing waves), for fixed $k_\phi > 0$.

Now $\tan \gamma = y/x$, where $x$ is the radial displacement from the corotation circle, and $y$ is the azimuthal displacement from the origin. From the background linear shear, it should be clear that $\tan \gamma = [y(0) + (3/2)\Omega xt]/x$. Note the sign convention we have chosen $(+3/2)$, which correctly implies that $\tan \gamma$ increases with time. Dividing through top and bottom by $x$ to find that $\tan \gamma = \tan \gamma(0) + (3/2) \Omega t$. Then we’re done: $k_r = k_\phi \tan \gamma = k_\phi(\tan \gamma(0) + (3/2)\Omega t) = k_r(0) + (3/2)\Omega k_\phi t$.

Waves that are initially leading eventually become trailing and ever more tightly wound, as $k_r$ increases towards ever larger positive values.

Figure 3: Wavefronts shearing in the shearing sheet.