# Astrophysical Fluid Dynamics – Problem Set 12

Readings: However much you like of Shu Chapter 8 on fluid instabilities and Chapter 10 on mixing length theory of convection

**Problem 1.** *Just How Tiny is the Tiny Superadiabatic Temperature Gradient?*

*Convection is often so efficient at transporting heat that the actual temperature gradient in a convective atmosphere is only slightly steeper than the adiabatic temperature gradient. Here we estimate quantitatively what "slightly steeper" means.*

*Recall that the Brunt-Vaisala (B-V) frequency is the frequency of buoyant vertical motions in an atmosphere, and is given by*

$$\omega_{B-V}^2 = \left[ \left.\frac{\partial T}{\partial z}\right|_{actual} - \left.\frac{\partial T}{\partial z}\right|_{adiabatic} \right] \frac{g}{T}, \tag{1}$$

*where $g > 0$ is the vertical gravitational acceleration, $T$ is temperature, and $z$ measures vertical height. Define*

$$\Delta\nabla T = \left[ \left.\frac{\partial T}{\partial z}\right|_{actual} - \left.\frac{\partial T}{\partial z}\right|_{adiabatic} \right] \tag{2}$$

*to be the difference between the actual temperature gradient and the adiabatic temperature gradient. We will estimate $\Delta\nabla T$, and compare it to $\nabla T|_{actual}$. Remember that if $\Delta\nabla T < 0$, then $\omega_{B-V}^2 < 0$— in other words, the B-V frequency is imaginary, any vertical motions are unstable, and convection ensues. Since $\nabla T < 0$, $\Delta\nabla T < 0$ means the absolute value of the actual temperature gradient, $|\nabla T|_{actual}$, exceeds the absolute value of the adiabatic temperature gradient, $|\nabla T|_{adiabatic}$; we say the actual temperature gradient is superadiabatic (but not by much) in convective atmospheres.*

*(a) [5 points] Consider a parcel of gas moving adiabatically upwards through a convective (superadiabatic) atmosphere. The parcel has mass density $\rho$ and specific heat [erg/(gram K)] at constant pressure $c_p$. It maintains pressure equilibrium with its surroundings: as it rises, the parcel's pressure matches exactly the surrounding atmospheric pressure (which is decreasing with increasing height).*

*The parcel's temperature decreases adiabatically, while the background atmosphere's temperature drops superadiabatically. In other words, the adiabatic drop in the parcel's temperature is not as much as the drop in the surrounding environment's temperature, because the actual temperature gradient of the environment is superadiabatic.*

*After the parcel has risen length $l$, where $l$ is small compared to the pressure scale height, how much excess energy density [erg/cm$^3$] does the parcel carry relative to its surroundings? Use the variables given above. Call this extra energy density $\epsilon$.*

*(b) [5 points] Give an approximate symbolic expression for the upward velocity, $v$, of the buoyant parcel after it has travelled distance $l$. Remember that the parcel is unstably buoyant; it experiences an upward acceleration, $(\delta\rho/\rho)g$, where $g$ is the local (downward) planetary gravitational acceleration. You should first understand why $\delta\rho$, the density difference between the parcel and its surroundings, is negative. Reduce your expression to one that does not contain $\rho$ or $\delta\rho$, but does contain $T$. Remember that pressure differences relative to the background are assumed zero, $\delta P = 0$.*

*(c) [5 points] Assume that convection dominates heat transport through the atmosphere. The atmosphere transports an energy flux $F$ [erg cm$^{-2}$ s$^{-1}$] upward (against gravity).*[1] *From the fisherman's mantra (density times speed equals flux), $F = \epsilon v$.*

*Use $F = \epsilon v$ and (a) and (b) to solve for an approximate symbolic expression for $\Delta\nabla T$. Your answer should depend on $F$, $l$, $T$, $g$, $c_p$, and $\rho$.*

*(d) [3 points] Calculate $|(\Delta\nabla T)/(\nabla T)_{actual}|$ for conditions appropriate to Jupiter's atmosphere at a pressure $P = 1$ bar. At this pressure, the temperature $T \approx 170\,\mathrm{K}$, $(\nabla T)_{actual} = -40\,\mathrm{K}/20\,\mathrm{km}$, $\mu m_{\mathrm{H}} \approx 4 \times 10^{-24}\,\mathrm{g}$, $g \approx 3 \times 10^{3}\,\mathrm{cm\,s^{-2}}$, $c_P \sim 10^8$ erg/g/K, $F \sim 6 \times 10^4$ erg/cm$^2$/s (comparable to the flux of sunlight absorbed by the planet).*[2]

*For this numerical evaluation, the only quantity which is not given by data is $l$, the distance a parcel travels before it dissolves away into its surroundings. No one knows what $l$—the infamous "mixing length" of convection—is. It is reasonable that $l < h$, the pressure scale height of the atmosphere. We charge forth boldly and use $l \sim h$.*

*Is your computed dimensionless quantity tiny?*

*(e) [5 points] Show that $F = \epsilon v \sim \rho v^3$, which requires that $\epsilon \sim \rho v^2$; in other words, a good fraction (not all) of the convective flux is in kinetic energy. You may find parts (a) and (b) helpful; also continue to use $l \sim h$.*

*The formula $F \sim \rho v^3$ is useful. It gives an estimate of the convective fluid velocities $v \sim (F/\rho)^{1/3}$ given $F$ and $\rho$.*

*(f) [2 points] Convection in combination with rotation can amplify seed magnetic fields (this is called a dynamo). Christensen et al. (2009, Nature) posited that the magnetic field grows until its energy density becomes comparable to the energy density of convective motions. This equipartition*

[1] In a star, the source of energy is fusion. In a planet, the source of energy can be radioactivity, the gravitational heat of accretion (a.k.a. heat of formation), radiation from the host star deposited at depth, or external tidal gravitational forces.

[2] Jupiter outputs about twice the energy it absorbs from the Sun. The extra energy derives from Jupiter's primordial heat of formation.

*argument leads to a scaling relation that seems to work approximately well over a wide range of scales, from Earth to Jupiter to sufficiently rapidly rotating Sun-like stars.*

*Using the energy density argument, derive a rough scaling relation for the magnitude of the magnetic field $B$ in terms of the convective flux $F$ and density $\rho$ (you may compare your answer to Christensen et al. 2009).*

**Problem 2.** *Gravity Waves and the Kelvin-Helmholtz Instability*

*Consider a fluid of density $\rho_+$ moving horizontally in the $x$-direction with constant speed $U_0$ above a second fluid of density $\rho_-$ which is at rest. The boundary between the two fluids is located at $z = 0$ and is idealized as having zero thickness (yes, a formal discontinuity!). Gravity points downward with constant acceleration $\vec{g} = -g\hat{z}$. We ignore viscosity and surface tension.*

*The density, pressure, and velocity fields read:*

$$\rho(x, z) = \rho_0(z) + \delta\rho(x, z, t) \tag{3}$$

$$P(x, z) = P_0(z) + \delta P(x, z, t) \tag{4}$$

$$\vec{u} = U_0 H(z)\hat{x} + \delta u(x, z, t)\hat{x} + \delta w(x, z, t)\hat{z} \tag{5}$$

*where $H(z)$ is the Heaviside function (+1 for $z > 0$ and 0 for $z < 0$). All quantities subscripted 0 denote background quantities, while $\delta$-quantities are perturbations. Note that the background $\rho_0$ is not a constant ($\rho_0 = \rho_+$ for $z > 0$ and $\rho_0 = \rho_-$ for $z < 0$), while $U_0$ is constant.*

*The mass and momentum equations read, as usual:*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{6}$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \nabla)\vec{u} = -\vec{\nabla} P + \rho \vec{g} \tag{7}$$

*The background is assumed to be in hydrostatic equilibrium. This problem determines under what conditions the perturbations are stable vs. unstable.*

*Much of the solution to this problem is contained in the Course Reader pages 293–297, reprinted from Chandrasekhar's book on hydrodynamics. You are free to use as much of the solution as you like (our notation differs somewhat from his; also our problem set-up is simpler). Section 14.6 of Thorne & Blandford may also be helpful.*

*(a) [2 points] Assume the flow is incompressible: $D\rho/Dt = 0$, where $D/Dt$ is the Lagrangian derivative. Prove that as a consequence $\nabla \cdot \vec{u} = 0$ (the flow is divergence-free).*

*(b) [15 points] Linearize the continuity equation and the two components of the momentum equation, considering perturbations of traveling wave form:*

$$\delta\rho, \delta P, \delta u, \delta w \propto \exp i(kx - \omega t) \tag{8}$$

*Note we are NOT YET specifying the $z$-dependence of the perturbations. As usual, $k$ is the (horizontal, not vertical) wavenumber and $\omega$ is the wave frequency.*

*Combine the linearized equations to derive:*

$$\begin{aligned}\rho_0[\omega - kU_0H(z)]k^2\delta w =& \frac{\partial}{\partial z}\left\{\rho_0[\omega - kU_0H(z)]\frac{\partial(\delta w)}{\partial z} + \rho_0 k\delta w\frac{\partial(U_0H(z))}{\partial z}\right\} \\ &- \frac{gk^2\delta w}{\omega - kU_0H(z)}\frac{\partial\rho_0}{\partial z}\end{aligned} \tag{9}$$

*You may compare with Chandrasekhar's equation (16) (we neglect surface tension so $T_s = 0$). Chandrasekhar's procedure, which you may follow, is to (i) combine the $x$-component of the momentum equation and $\nabla \cdot \vec{u} = 0$ to derive an equation for $\delta P$ in terms of various $z$-derivatives (cf. his equation 14), (ii) combine the $z$-component of the momentum equation and continuity to derive an equation for $\delta w$ in terms of various $z$-derivatives (cf. his equation 15), and (iii) combine (i) and (ii) to eliminate $\delta P$.*

*(c) [3 points] Write down (9) AWAY from the boundary $z = 0$. Do this for $z > 0$, and repeat for $z < 0$, using the fact that $\rho_0 = \rho_+ =$ constant for $z > 0$, and $\rho_0 = \rho_- =$ constant for $z < 0$. Show that in either regime,*

$$k^2\delta w = \frac{\partial^2}{\partial z^2}\delta w \tag{10}$$

*Physically sensible solutions that decay to zero far away from the boundary are:*

$$\delta w = \delta w_+ \exp -|k|z \quad \text{for } z > 0 \tag{11}$$

$$\delta w = \delta w_- \exp +|k|z \quad \text{for } z < 0 \tag{12}$$

*(NB: the coefficients $\delta w_+$ and $\delta_-$ are functions of $x$ and $t$ following (8). As the next part will show, $\delta w_+ \neq \delta w_-$.)*

*(d) [5 points] Now we examine the boundary between the two fluids. Though we may have implied the boundary is located at $z = 0$, we recognize that the boundary does not literally and strictly equal $z = 0$ at all times; if it did, there would be no perturbation and this would be a very boring problem. Rather, the boundary, defined as the interface between the two fluids $\rho_+$ and $\rho_-$, can change with position $x$ and time $t$ as the perturbed upper fluid reaches downward and the perturbed lower fluid reaches upward (without ever mixing). Chandrasekhar refers to the boundary as being located at some $z = z_s(x,t)$, where $z_s \simeq 0$ since the fluid displacements are infinitesimal (but not strictly zero).*

*The boundary moves with a vertical perturbation velocity $\delta w(z_s) \equiv D(\delta z_s)/Dt$, where $\delta z_s$ is the vertical displacement of a fluid parcel exactly at the interface. We can, if we wish, picture such an interfacial fluid parcel as being filled with $\rho_+$ fluid in its upper half, and filled with $\rho_-$ fluid in its lower half. Like all other perturbed quantities in this problem, $\delta z_s \propto \exp i(kx - \omega t)$ (the interface is crinkled in the form of a traveling wave).*

*Now consider another fluid parcel just below this interface parcel. Say it is vertically displaced by $\delta z_-$. Similarly, just above the interface parcel, another fluid parcel is vertically displaced by $\delta z_+$.*

*The key boundary condition is that fluid displacements are continuous across the interface: $\delta z_- = \delta z_s = \delta z_+$ in an infinitesimal interval around $z_s$.*

*Use $\delta w = D(\delta z)/Dt$ and $\delta z_- = \delta z_+$ across the boundary to show that $\delta w/[\omega - kU_0 H(z)]$ is continuous across the boundary. That is, show that*

$$\frac{\delta w_+}{\omega - kU_0} = \frac{\delta w_-}{\omega} \tag{13}$$

*(e) [15 points] Integrate (9) across the boundary from $z = z_s - \epsilon$ to $z = z_s + \epsilon$, and take the limit $\epsilon \to 0$. Thereby derive:*

$$0 = (\rho_- + \rho_+)\omega^2 - 2\rho_+ kU_0\omega + \left[\rho_+(kU_0)^2 - gk(\rho_- - \rho_+)\right] \tag{14}$$

*You will need to use (11)–(12) to evaluate the $z$-derivatives, and (13).*

*Define $\alpha_- \equiv \rho_-/(\rho_- + \rho_+)$ and $\alpha_+ \equiv \rho_+/(\rho_- + \rho_+)$, and solve for the dispersion relation*

$$\omega = \alpha_+ kU_0 \pm \sqrt{gk(\alpha_- - \alpha_+) - \alpha_-\alpha_+ k^2 U_0^2} \tag{15}$$

*(f) [3 points] When the bottom fluid is water, the top fluid is air (whose density is $\ll$ than water), and $U_0 = 0$ (no shear), this dispersion relation reduces to $\omega \simeq \sqrt{gk}$. This is the well-known dispersion relation for "deep water waves". The restoring force is gravity; hence these are also called "gravity waves".[3] Here "deep" refers to the fact that the horizontal wavelengths are short compared to the depth of the water. Gravity waves are commonplace in nature—in stably stratified (non-convecting) atmospheres, and stellar and planetary interiors. In stars, "g-modes" are standing gravity waves in the radiative (non-convecting) portions of stars (as distinct from "p-modes" which are standing pressure waves, a.k.a. sound waves).*

*Solve for the group velocity $d\omega/dk$ for deep water waves in terms of $g$ and $k$. Do long-wavelength waves or short-wavelength waves travel faster? Waves created in an ocean storm disperse according to their wavelength; do the long-wavelength waves or the short-wavelength waves arrive at the shore first?*

*(g) [2 points] From the dispersion relation derive the well-known result that perturbations are Kelvin-Helmholtz unstable if*

$$k > \frac{g(\alpha_- - \alpha_+)}{\alpha_+\alpha_- U_0^2} \tag{16}$$

[3] Not to be confused with "gravitational waves" which are fluctuations in the fabric of space itself.

*i.e. for sufficiently short-wavelength disturbances. Notice* $U_0^2$ *is in the denominator while* $g$ *is in the numerator; gravity stabilizes disturbances as long as* $\alpha_- > \alpha_+$ *(denser fluid is on the bottom), while larger* $U_0$ *supplies more free energy to destabilize the flow. Notice also if* $\alpha_- < \alpha_+$ *(denser fluid overlaying lighter fluid), modes of all wavelengths are unstable — this is the Rayleigh-Taylor instability at work (why the heavy cream in Thai ice tea never stays on top).*