Astrophysical Fluid Dynamics – Problem Set 12

Readings: Course Reader selections from Binney & Tremaine on Toomre $Q$, and from Frank, King, & Raine on accretion disks

**Problem 1.** Toomre $Q$ Mnemonics

A gas disk orbits an object of mass $M$. The disk is in vertical hydrostatic equilibrium with thickness $h$. You may wish to refer back to PS 1, Problem 3d where you solved for vertical hydrostatic equilibrium in a disk. You may use the results of that problem here, even though that earlier problem neglected disk self-gravity when calculating $h$; the error in using that $h$ is on the order of unity for $Q \sim 1$ (and smaller for $Q \gg 1$).

(a) [4 points] Show that $Q \equiv c_s \Omega / (\pi G \Sigma) \sim 1$ at radius $r$ corresponds to the condition $\rho \sim \rho_{\text{crit}} \sim M / r^3$, where $\rho$ is the gas density (mass per unit volume) at the disk midplane. Here $c_s$ is the gas sound speed, $G$ is the gravitational constant, $\Sigma$ is the mass per unit area of the disk, and $\Omega$ is the angular frequency, where all variables are evaluated locally at $r$.

The condition you have derived is an easy way to remember the Toomre instability condition. The critical density $\rho_{\text{crit}}$ is sometimes called the Roche density (and is just the mass of the central object divided into the sphere just enclosing your position).

(b) [3 points] Show that for $Q \sim 1$, the marginally unstable mode has a radial wavelength comparable to the gas vertical scale height $h$.

(c) [3 points] It is sometimes stated that $Q < 1$ corresponds to $M_{\text{disk}} / M > h / r$. Explain why this statement makes rough sense.

**Problem 2.** Alpha Disks

An accretion disk having surface density $\sigma(r)$ at radius $r$ orbits a point mass $M$. The potential is Keplerian so the disk angular frequency $\Omega \propto r^{-3/2}$.

(a) [3 points] In class we had used the $\hat{\phi}$ component of the momentum equation to see that the radial accretion velocity $u_r$ is given by:

$$\sigma u_r = \frac{\partial}{\partial r} \left( \frac{r^2 \int \tau_{r\phi} dz}{r^2 \left( \frac{\partial (r \Omega)}{\partial r} + \Omega \right)} \right).$$  

(1)

Model the stress in terms of a shear viscosity $\nu$:

$$\tau_{r\phi} = \rho \nu r \frac{\partial \Omega}{\partial r}.$$  

(2)

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1Technically the Toomre $Q$ contains not the angular frequency $\Omega$, but the radial epicyclic frequency $\kappa$. But the two frequencies are the same to within factors of a few; see the textbook by Binney & Tremaine.
and derive the mass accretion rate (the mass crossing a circle of radius \( r \) per unit time):

\[
\dot{M} \equiv -2\pi \sigma u_r r = +6\pi \left[ r \frac{\partial (\sigma \nu)}{\partial r} + \frac{\sigma \nu}{2} \right]
\]  

(3)

where \( \sigma = \int \rho \, dz \). We have inserted a minus sign in the definition of \( \dot{M} \) since astronomers like to think of \( \dot{M} > 0 \) when mass flows radially in. If \( \dot{M} < 0 \), we have a “decretion” disk where mass flows radially outward. In general, \( \dot{M} \) can vary with radius, causing mass to pile up or drain out locally in a non-steady evolution.

(b) [3 points] For the remaining parts of this problem, assume steady accretion: \( \dot{M} \) is constant with radius (so mass does not pile up anywhere and \( \partial \sigma / \partial t = 0 \); you can see this last statement from the continuity equation we wrote down in class). From (3) derive:

\[
\dot{M} = 3\pi \sigma \nu.
\]  

(4)

Hint: Solve for \( y \equiv \sigma \nu \) in (3) and assume that \( \sigma \nu \) is finite at \( r = 0 \).

(c) [3 points] Show that the result in (b) can be intuited to order-of-magnitude by taking the disk mass \( M_{\text{disk}} \sim \sigma r^2 \) and the diffusion time \( t \sim r^2 / \nu \). Show also that the radial accretion speed \( u_r \sim \nu / r \).

(d) [3 points] Accretion disks get hot. This is true whether we model them as viscous (“honey”) or turbulent. For a gas parcel of mass \( \Delta m \) to go from a circular orbit at \( r_1 \) to a circular orbit at \( r_2 < r_1 \), it has to lose orbital energy (in our Kepler potential, the total orbital energy is \( -GM\Delta m / (2r) \), so smaller \( r \) implies lower energy). That energy goes into heating the disk.

The heating rate per unit area is given very nearly by

\[
D = \frac{3}{4\pi} \frac{GM \dot{M} r^3}{r^3}
\]  

(5)

a result that can be derived in either the viscous picture (as is done in the Frank, King, & Raine book chapters reprinted in the Course Reader) or the turbulent picture (Balbus, Gammie, & Hawley 1994, MNRAS, 271). To order-of-magnitude, this result also makes sense: the energy lost by \( \Delta m \) in spiralling from \( r \) to \( r/2 \) is \( \sim GM\Delta m / r \) (dropping all 2’s); delivering \( \Delta m \) per time \( \Delta t \) means we are dissipating a power \( \sim (GM\Delta m / r) / \Delta t \sim GM \dot{M} / r \); and spreading this power over an annulus of area \( \sim r^2 \) gives \( \sim GM \dot{M} / r^3 \), as in (5) above.

If the disk is able to radiate away this heat as a blackbody,\(^2\) then we have

\[
D = 2\sigma_{\text{SB}} T^4
\]  

(6)

or

\[
\sigma_{\text{SB}} T^4 = \frac{3}{8\pi} \frac{GM \dot{M} r^3}{r^3}
\]  

(7)

\(^2\)It doesn’t have to; it can radiate as a non-blackbody, or it might not radiate at all. If the latter, the accretion flow is called “advection-dominated”—the energy goes into heating the gas and just gets swept along radially. Advection-dominated accretion flows tend to be vertically thick/puffy.
where $\sigma_{SB}$ is the Stefan-Boltzmann constant. The factor of 2 in (6) accounts for the top and bottom faces of the disk, both of which radiate into space.\(^3\)

From (7), the assumption of a steady disk, and the assumption that the disk is vertically isothermal,\(^4\) decide how the vertical disk scale height $h$ scales with $r$. Sketch how the disk looks.

(e) [2 points] Now assume the kinematic viscosity $\nu = \alpha c_s h$, where $c_s$ is the sound speed, and $\alpha$ is a dimensionless factor, assumed to be a strict constant. How does $\sigma$ scale with $r$?

(f) [2 points] Suppose there is a pressure disturbance that perturbs the disk out of vertical hydrostatic equilibrium. Find an approximate expression for how long it takes this disturbance to propagate away vertically. Call this the dynamical time $t_z$, and express in terms of $\Omega$.

(g) [4 points] Find an approximate expression for how long it takes a temperature disturbance to equilibrate away. That is, consider perturbing the temperature so that $T \to T + \delta T$. The heat content of the disk has increased, but so has the blackbody flux to space. It takes some time for the perturbation flux to carry away the perturbation heat. Call this the cooling time $t_{\text{cool}}$, and express in terms of $\Omega$ and $\alpha$.

(h) [3 points] Find an approximate expression for how long it takes a mass disturbance—say a local bunching of material on a radial length scale $r$—to diffuse radially away. Call this the viscous time $t_{\text{visc}}$, and express in terms of $\Omega$, $\alpha$, and $h/r$.

Arrange $t_z$, $t_{\text{cool}}$ and $t_{\text{visc}}$ in increasing order, assuming $\alpha < 1$.

(i) [2 points] How does Toomre’s $Q$ scale with radius $r$? Accretion disks tend to be gravitationally unstable at large radii, leading some to surmise that the outer peripheries of quasar accretion disks / protostellar accretion disks are fertile breeding grounds for starbursts / binary companion stars or brown dwarfs.\(^5\)

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\(^3\)Side A and Side B of the vinyl record. (Do people still play vinyl?)

\(^4\)A disk that is optically thick will not be vertically isothermal. It will be hotter at the midplane where most of the disk mass resides and where energy dissipation rates are highest. We are neglecting this vertical temperature gradient.

\(^5\)While wide stellar binaries are thought to form by gravitational instability, the story must be different for compact stellar binaries (having separations less than 30 AU or so) which cannot have formed by gravitational instability where they are. Compact binaries are thought to start off wide—that is, they form just like wide binaries by gravitational instability—and then have their orbits hardened (made smaller) by interacting with residual disk gas. In other words, the disk forces the binary companion to migrate inward (via a form of dynamical friction).