Astrophysical Fluid Dynamics – Problem Set 12

Readings: However much of the hand-outs by Balbus & Hawley that you like; Shu 93–98, 101–110, 120–123, 125–130

Problem 1. Springtime for MRI[1], or "Saving Astronaut Silverman"

Consider a body (space shuttle) in circular orbit about another (Earth) with angular frequency Ω_0 and radius R_0. Make the simplifying assumption that the potential is purely Keplerian.

Attached to the orbiting body is a small test mass (astronaut). They are attached by a spring, and orbit in the same plane. The spring can extend in either the radial r or azimuthal ϕ directions, by amounts ξ_r and ξ_ϕ, both of which are much smaller than R_0 in magnitude (the spring is short). The elastic force on the test mass is $-k\vec{\xi}$, where k is the spring constant.

Determine the condition on k such that the spring either (a) stretches indefinitely (instability), or (b) oscillates (stability). Neglect the back-reaction of the test mass on the orbiting body.

You can solve this problem any way you like, but you might like to start from the equations of motion for the test mass in an inertial (non-rotating) frame:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{d\Phi}{dr} - k\xi_r/m \tag{1}$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = -k\xi_\phi/m \tag{2}$$

where Φ is the gravitational potential and m is the test mass. If you say that $\vec{\xi} = \vec{\xi_0}\exp(i\omega t)$, you can derive an equation for ω that should bear an astonishing resemblance to the dispersion relation derived in class for the MRI.

Problem 2. Just How Tiny is the Tiny Superadiabatic Temperature Gradient?

It is often stated that convection is so efficient at transporting heat that the actual temperature gradient in a convective atmosphere is only slightly steeper than the adiabatic temperature gradient. Here we estimate quantitatively what "slightly steeper" means.

Recall that the Brunt-Vaisala (B-V) frequency is the frequency of buoyant vertical motions in an atmosphere, and is given by

$$\omega_{B-V}^2 = \left[\left.\frac{\partial T}{\partial z}\right|_{actual} - \left.\frac{\partial T}{\partial z}\right|_{adiabatic} \right] \frac{g}{T}\,, \tag{3}$$

[1]It has been said that you can make a good career in physics using only the simple harmonic oscillator.

where $g > 0$ is the gravitational acceleration, T is temperature, and z is height above the base of the atmosphere. Define

$$\Delta\nabla T = \left[\left.\frac{\partial T}{\partial z}\right|_{actual} - \left.\frac{\partial T}{\partial z}\right|_{adiabatic} \right] \tag{4}$$

to be the difference between the actual temperature gradient and the adiabatic temperature gradient. We will estimate $\Delta\nabla T$, and compare it to $\nabla T|_{actual}$. Remember that if $\Delta\nabla T < 0$, then $\omega^2_{B-V} < 0$—in other words, the B-V frequency is imaginary, any vertical motions are unstable, and convection ensues. Since $\nabla T < 0$, $\Delta\nabla T < 0$ means the *absolute value* of the actual temperature gradient, $|\nabla T|_{actual}$, *exceeds* the *absolute value* of the adiabatic temperature gradient, $|\nabla T|_{adiabatic}$; we say the actual temperature gradient is *superadiabatic* (but not by much) in convective atmospheres.

(a) Consider a parcel of gas moving *adiabatically* upwards through a convective (superadiabatic) atmosphere. The parcel has mass density ρ and specific heat [erg/(gram K)] at constant pressure c_p. It maintains pressure equilibrium with its surroundings: as it rises, the parcel's pressure matches exactly the surrounding atmospheric pressure (which is decreasing with increasing height). The parcel's temperature decreases adiabatically, while the atmosphere's temperature drops superadiabatically. In other words, the adiabatic drop in the parcel's temperature is not as much as the drop in the surrounding environment's temperature, because the actual temperature gradient of the environment is superadiabatic.

After the parcel has risen length l, where l is small compared to the pressure scale height, how much *excess* energy density [erg/cm^3] does the parcel carry relative to its surroundings? Use the variables given above. Call this extra energy density ϵ.

(b) Give an approximate symbolic expression for the upward velocity, v, of the buoyant parcel after it has travelled distance l. Remember that the parcel is unstably buoyant; it experiences an upward acceleration, $(\delta\rho/\rho)g$, where g is the local (downward) planetary gravitational acceleration. You should first understand why $\delta\rho$, the density difference between the parcel and its surroundings, is negative. Reduce your expression to one that does not contain ρ or $\delta\rho$, but does contain T.

(c) Assume that convection dominates heat transport through the atmosphere. The atmosphere must transport the full flux of energy that is deposited at its base. Call this flux, F [erg cm^{-2} s^{-1}]. Therefore $F = \epsilon v$ (the density of any quantity times the speed with which that quantity moves gives you a flux).

Use $F = \epsilon v$ and (a) and (b) to solve for an approximate symbolic expression for $\Delta\nabla T$. Your answer should depend on F, l, T, g, c_p, and ρ.

(d) Calculate $|(\Delta\nabla T)/(\nabla T)_{actual}|$ for conditions appropriate to Jupiter's atmosphere at a pressure $P = 1$ bar. At this pressure, the temperature $T \approx 170\,\mathrm{K}$, $(\nabla T)_{actual} = -40\,\mathrm{K}/20\,\mathrm{km}$, $\mu \approx 4 \times 10^{-24}\,\mathrm{g}$, and $g \approx 3 \times 10^3\,\mathrm{cm\,s^{-2}}$.

For this numerical evaluation, the only quantity which is not given by data is l, the distance a parcel travels before it dissolves away into its surroundings. No one knows what l—the infamous "mixing length" of convection—is. It is reasonable that $l < h$, the pressure scale height of the atmosphere. We charge forth boldly and use $l \sim h$.

For F, remember that the energy that is transported upwards through the atmosphere is absorbed energy from the Sun, times a factor of $\sim$2 to account for the internal heat that Jupiter self-generates because it is still (after all these eons) gravitationally contracting and converting gravitational potential energy into heat. Take the (Bond) albedo of Jupiter to be 0.5 when computing the flux absorbed.

Is your computed dimensionless quantity tiny?