Readings: Course Reader selections from Binney & Tremaine on Toomre’s $Q$, and from Frank, King, & Raine on accretion disks

Problem 1. Toomre $Q$ Mnemonics

A gas disk orbits an object of mass $M$. The disk is in vertical hydrostatic equilibrium with thickness $h$. You may wish to refer back to PS 1, Problem 3d where you solved for vertical hydrostatic equilibrium in a disk. You may use the results of that problem here, even though that earlier problem neglected disk self-gravity when calculating $h$; the error in using that $h$ is on the order of unity for $Q \sim 1$ (and smaller for $Q \gg 1$).

(a) [4 points] Show that, to order-of-magnitude, $Q \equiv c_s \Omega / (\pi G \Sigma) \sim 1$ at radius $r$ corresponds to the condition $\rho \sim \rho_{\text{crit}} \sim M/r^3$, where $\rho$ is the gas density (mass per unit volume) at the disk midplane. Here $c_s$ is the gas sound speed, $G$ is the gravitational constant, $\Sigma$ is the mass per unit area of the disk, and $\Omega$ is the angular frequency, where all variables are evaluated locally at $r$.\footnote{Technically the Toomre $Q$ contains not the angular frequency $\Omega$, but the radial epicyclic frequency $\kappa$. But the two frequencies are the same to within an order-unity factor; see the textbook by Binney & Tremaine, or the problem set question on radial epicycles and Rayleigh stability.}

The condition you have derived is an easy way to remember the Toomre instability condition. The critical density $\rho_{\text{crit}}$ is sometimes called the Roche density (and is just the mass of the central object divided into the sphere just enclosing your position).

We derived in lecture that the Toomre $Q$ for a gas disk is given by

$$Q = \frac{c_s \kappa}{\pi G \Sigma}. \quad (1)$$

We substitute the hydrostatic vertical scale height of the disk, given by $h = c_s / \Omega$ (Problem Set 1) for $c_s$ and approximate the epicyclic frequency by $\kappa \sim \Omega$. Note that $\rho \sim \Sigma / (2h)$, where $\Sigma$ is the disk surface mass density. We obtain the following expression

$$Q \sim \frac{\Omega^2}{2\pi G \rho} < 1. \quad (2)$$

Assuming Keplerian orbits,

$$\Omega = \sqrt{\frac{GM}{r^3}}. \quad (3)$$

If we substitute the Keplerian value for $\Omega$ into equation (2) and solve for $\rho$ we obtain the desired relation

$$\boxed{\rho > \rho_{\text{crit}} \sim \frac{M}{2\pi r^3}} \quad (4)$$
(b) [3 points] Show that for $Q \sim 1$, the marginally unstable mode has a radial wavelength that scales as the gas vertical scale height $h$.

We derived in class that when $Q = 1$, the wavenumber that goes unstable (and the fastest growing mode when $Q < 1$) is

$$k_{\text{crit}} = \frac{\pi G \Sigma}{c_s^2} \quad (5)$$

Equivalently the radial wavelength $\lambda_{\text{crit}} = \frac{2\pi}{k_{\text{crit}}}$ equals

$$\lambda_{\text{crit}} = \frac{2c_s^2}{G \Sigma} \quad (6)$$

But when $Q = 1$, $G \Sigma = c_s \kappa / \pi = c_s \Omega / \pi$ which when substituted into the above gives:

$$\lambda_{\text{crit}} = 2\pi c_s / \Omega = 2\pi h$$

as desired.

(c) [3 points] It is sometimes stated that $Q < 1$ corresponds to $M_{\text{disk}}/M > h/r$. Explain why this statement makes rough sense.

In equation (4) we derived that for $Q < 1$

$$\rho \sim \frac{\Sigma}{2h} > \frac{M}{2\pi r^3}. \quad (8)$$

The statement makes sense if you approximate the local disk mass as

$$M_{\text{disk}} \sim \Sigma \pi r^2$$

(9)

because then the inequality (8) reduces to

$$M_{\text{disk}}/M > h/r$$

Problem 2. Alpha Disks

An accretion disk having surface density $\sigma(r)$ at radius $r$ orbits a point mass $M$. The potential is Keplerian so the disk angular frequency $\Omega \propto r^{-3/2}$.

(a) [3 points] In class we had used the $\hat{\phi}$ component of the momentum equation to see that the radial accretion velocity $u_r$ is given by:

$$\sigma u_r = \frac{\partial}{\partial r} \left( \frac{r^2 \int \tau_{r\phi} dz}{r^2 \left( \frac{\partial (r \Omega)}{\partial r} + \Omega \right)} \right) \quad (10)$$

Model the stress in terms of a shear viscosity $\nu$:

$$\tau_{r\phi} = \rho \nu \frac{\partial \Omega}{\partial r}$$

(11)

and derive the mass accretion rate (the mass crossing a circle of radius $r$ per unit time):

$$\dot{M} = -2\pi \sigma u_r r = +6\pi \left[ r \frac{\partial (\sigma \nu)}{\partial r} + \frac{\sigma \nu}{2} \right]$$

(12)
where $\sigma = \int \rho \, dz$. We have inserted a minus sign in the definition of $\dot{M}$ since astronomers like to say $\dot{M} > 0$ when mass flows radially in. If $\dot{M} < 0$, we have a “decretion” disk where mass flows radially outward. In general, $\dot{M}$ can vary with radius, causing mass to pile up or drain out locally in a non-steady evolution.

First evaluate the denominator in (10) using the assumption of Keplerian orbits:

$$\sigma u_r = \frac{2}{\Omega r^2} \frac{\partial}{\partial r} \left( r^2 \int \tau \phi \, dz \right)$$  \hspace{1cm} (13)

Then insert (24) and use Keplerian orbits again:

$$\sigma u_r = \frac{2}{\Omega r^2} \frac{\partial}{\partial r} \left( r^2 \int \rho \nu r \left( -\frac{3\Omega}{2r} \right) \, dz \right)$$  \hspace{1cm} (14)

Take $\nu$ and $\Omega$ to be constant with $z$:

$$\sigma u_r = -\frac{3}{\Omega r^2} \frac{\partial}{\partial r} \left( r^2 \nu \sigma \right)$$  \hspace{1cm} (15)

so

$$\sigma u_r = -3 \frac{\partial}{\partial r} \left( \nu \sigma \right) - \nu \frac{3}{2r}$$  \hspace{1cm} (16)

Finally plug into the definition of $\dot{M}$ to find the desired result.

(b) [3 points] For the remaining parts of this problem, assume steady accretion: $\dot{M}$ is constant with radius (so mass does not pile up anywhere and $\partial \sigma / \partial t = 0$; you can see this last statement from the continuity equation we wrote down in class). From (10) derive:

$$\dot{M} = 3\pi \sigma \nu .$$  \hspace{1cm} (17)

Hint: Solve for $y \equiv \sigma \nu$ in (12) and assume that $\sigma \nu = \text{finite at} \ r = 0$.

Partial spatial derivatives become full spatial derivatives because we are assuming steady conditions (no time dependence). Do as the hint says and re-write (12) as:

$$\frac{\dot{M}}{6\pi} - \frac{y}{2} = r \frac{dy}{dr}$$  \hspace{1cm} (18)

This is separable:

$$\frac{dy}{\dot{M}/(6\pi) - y/2} = \frac{dr}{r}$$  \hspace{1cm} (19)

and integrates to:

$$-2 \log \left( \frac{\dot{M}/(6\pi) - y/2}{2} \right) = \log r + C_0$$  \hspace{1cm} (20)

Exponentiating both sides:

$$\left( \frac{\dot{M}/(6\pi) - y/2}{2} \right)^{-2} = C_1 r$$  \hspace{1cm} (21)

which re-arranges to:

$$y = \frac{\dot{M}}{3\pi} - C_2 / \sqrt{r}$$  \hspace{1cm} (22)
If \( y = \sigma \nu = \text{finite at } r = 0 \), we must have \( c_2 = 0 \). Then: \( \dot{M} = 3\pi \sigma \nu \) as desired.

(c) [3 points] Show that the result in (b) can be intuited to order-of-magnitude by taking the disk mass \( M_{\text{disk}} \sim \sigma r^2 \) and the diffusion time \( t \sim r^2/\nu \). Show also that the radial accretion speed \( u_r \sim \nu/r \).

Take \( \dot{M} \sim M_{\text{disk}}/t \sim \sigma r^2/(\nu^2/\nu) \sim \sigma \nu \) as desired. Combine this result with \( \dot{M} \sim \sigma u_r r \) to find \( u_r \sim \dot{M}/(\sigma r) \sim (\sigma \nu)/(\sigma r) \sim \nu/r \) as desired.

(d) [3 points] Accretion disks get hot. This is true whether we model them as viscous (“honey”) or turbulent. For a gas parcel of mass \( \Delta m \) to go from a circular orbit at \( r_1 \) to a circular orbit at \( r_2 < r_1 \), it has to lose orbital energy (in our Kepler potential, the total orbital energy is \( -GM\Delta m/(2r) \), so smaller \( r \) implies lower energy). That energy goes into heating the disk. The heating rate per unit area is given very nearly by

\[
D = \frac{3}{4\pi} \frac{GM\dot{M}}{r^3} \tag{23}
\]

a result that can be derived in either the viscous picture (as is done in the Frank, King, & Raine book chapters reprinted in the Course Reader) or the turbulent picture (Balbus, Gammie, & Hawley 1994, MNRAS, 271). To order-of-magnitude, this result also makes sense: the energy lost by \( \Delta m \) in spiralling from \( r \) to \( r/2 \) is \( \sim GM\Delta m/r \) (dropping all 2’s); sending in \( \Delta m \) per time \( \Delta t \) means we are dissipating a power \( \sim (GM\Delta m/r)/\Delta t \sim GM\dot{M}/r \); and spreading this power over an annulus of area \( \sim r^2 \) gives \( \sim GM\dot{M}/r^3 \), as in (23) above.

If the disk is able to radiate away this heat as a blackbody,\(^2\) then we have

\[
D = 2\sigma_{\text{SB}} T^4 \tag{24}
\]

or

\[
\sigma_{\text{SB}} T^4 = \frac{3}{8\pi} \frac{GM\dot{M}}{r^3} \tag{25}
\]

where \( \sigma_{\text{SB}} \) is the Stefan-Boltzmann constant. The factor of 2 in (24) accounts for the top and bottom faces of the disk, both of which radiate into space.\(^3\)

From (25), the assumption of a steady disk, and the assumption that the disk is vertically isothermal,\(^4\) decide how the vertical disk scale height \( h \) scales with \( r \). Sketch how the disk looks.

Using \( c_s \propto \sqrt{T} \) and the fact that the power emitted per unit area (derived in lecture) is given by

\[
\sigma_{\text{SB}} T_{\text{eff}}^4 = \frac{3}{8\pi} \frac{GM_{\ast}\dot{M}}{r^3}, \tag{26}
\]

\(^2\)It doesn’t have to; it can radiate as a non-blackbody, or it might not radiate at all. If the latter, the accretion flow is called “advection-dominated”—the energy goes into heating the gas and just gets swept along radially. Advection-dominated accretion flows tend to be vertically thick/puffy.

\(^3\)Side A and Side B of the vinyl record. (Do people still play vinyl?)

\(^4\)A disk that is optically thick will not be vertically isothermal. It will be hotter at the midplane where most of the disk mass resides and where energy dissipation rates are highest. We are neglecting this effect.
we conclude that \( c_s \propto r^{-3/8} \), if \( T = T_{\text{eff}} \). Finally, from \( h \propto c_s/\Omega \) and assuming Keplerian orbits, \( \Omega \propto r^{-3/2} \), we get \( h \propto r^{-3/8 + 3/2} \propto r^{9/8} \).

\[
\sigma \propto r^{-3/4}
\]  

(27)

Figure 1: Section of an accretion disk (not to scale)

(e) [2 points] Now assume the kinematic viscosity \( \nu = \alpha c_s h \), where \( c_s \) is the sound speed, and \( \alpha \) is a dimensionless factor, assumed to be a strict constant. How does \( \sigma \) scale with \( r \)?

We derived above that

\[
\dot{M} = 3\pi \sigma \nu.
\]  

(28)

Under steady conditions, \( \dot{M} \) is constant and therefore \( \sigma \propto 1/\nu \). Setting \( \nu = \alpha c_s h \propto r^{-3/8} r^{9/8} \propto r^{3/4} \). Therefore:

\[
\sigma \propto r^{-3/4}
\]  

(29)

(f) [2 points] Suppose there is a pressure disturbance that perturbs the disk out of vertical hydrostatic equilibrium. Find an approximate expression for how long it takes this disturbance to propagate away vertically. Call this the dynamical time \( t_z \), and express in terms of \( \Omega \).

The pressure disturbance gets evened out in the time it takes a pressure wave to propagate vertically upward to vacuum (the shortest lengthscale to travel to reach a different pressure). The propagation speed is just the sound speed \( c_s \). The length scale is \( h \). So \( t_z \sim h/c_s \sim 1/\Omega \), also called the dynamical time.

(g) [4 points] Find an approximate expression for how long it takes a temperature disturbance to equilibrate away. That is, consider perturbing the temperature so that \( T \to T + \delta T \). The heat content of the disk has increased, but so has the blackbody flux to space. It takes some time for the perturbation flux to carry away the perturbation heat. Call this the cooling time \( t_{\text{cool}} \), and express in terms of \( \Omega \) and \( \alpha \).

Imagine the disk has temperature \( T + \delta T \), where \( T \) is the equilibrium temperature and \( \delta T \) its perturbation. The thermal energy per unit area of the disk is

\[
E + \delta E = \frac{3}{2} \mu k (T + \delta T)
\]  

(30)
(where $\delta E$ is the extra thermal energy in a vertical column through the disk). The power emitted per unit area is

$$F + \delta F = \sigma_{sb}(T + \delta T)^4 \approx \sigma_{sb}T^4 + 4\sigma_{sb}T^3\delta T$$

(31)

Since the equilibrium values balance each other out, we have

$$\frac{\delta E}{t_{cool}} \sim \delta F \sim 4\sigma_{sb}T^3\delta T.$$ 

(32)

Solving for $t_{cool}$ we get

$$t_{cool} \sim \frac{3}{8} \frac{\Sigma k}{\mu \sigma_{sb}T^3}$$

(33)

Since $c_s = \sqrt{\gamma kT/\mu}$, and using equation (26) to substitute the value of $\sigma_{sb}T^4$, we get

$$t_{cool} \sim \frac{\Sigma c_s^2 r^3}{GMM}.$$ 

(34)

If we now substitute the value for $\dot{M}$ given by equation (28) and set $\nu = c_s h \alpha$ we get the following expression

$$t_{cool} \sim \frac{c_s}{h \alpha GM} r^3.$$ 

(35)

Note that $r^3/(GM) = 1/(\Omega^2)$ and again $h = c_s / \Omega$; therefore

$$t_{cool} \sim \frac{1}{\alpha \Omega}.$$ 

(36)

(h) [3 points] Find an approximate expression for how long it takes a mass disturbance—say a local bunching of material on a radial length scale $r$—to diffuse radially away. Call this the viscous time $t_{visc}$, and express in terms of $\Omega$, $\alpha$, and $h/r$.

Arrange $t_z$, $t_{cool}$ and $t_{visc}$ in increasing order, assuming $\alpha < 1$.

This is like the time it takes a drop of honey to smear itself thin, so $t_{visc} \sim r^2/\nu$ like any good diffusion process. We substitute $\nu = c_s h \alpha$ and $c_s = h \Omega$ to obtain

$$t_{visc} \sim \frac{1}{\alpha} \left( \frac{r}{h} \right)^2 \frac{1}{\Omega}.$$ 

(37)

Since $0 < \alpha < 1$ and $0 < (h/r) < 1$

$$t_z < t_{cool} < t_{visc}.$$ 

(38)

(i) [2 points] How does Toomre’s $Q$ scale with radius $r$? Accretion disks tend to be gravitationally unstable at large radii, leading some to surmise that the outer peripheries of quasar accretion disks / protostellar accretion disks are fertile breeding grounds for starbursts / binary companion stars or brown dwarfs.5

While wide stellar binaries are thought to form by gravitational instability, the story must be more complicated for compact stellar binaries (having separations less than 30 AU or so). Compact binaries are thought to start off wide, and then have their orbits shrunken—hardened is the technical term—by interacting with disk gas. In other words, the disk forces the binary companion to migrate inward.
Toomre’s $Q \equiv c_s\kappa/(\pi G \Sigma)$. Substitute $\kappa = \Omega \propto r^{-3/2}$; use $c_s \propto r^{-3/8}$ and $\Sigma \propto r^{-3/4}$; then find $Q \propto r^{-3/8 + 3/2 + 3/4} \propto r^{-9/8}$ and conclude that the largest radii are the most prone to gravitational instability.