Problem 1. Magic for Astronomers

[10 points] Fill a tall glass with water, place a flat card covering the entire top of the glass, and upend the glass. Watch with amazement as the card remains suspended and water does not pour out!

Explain this phenomenon. What is pushing against the card to keep the water from falling? What is the tallest glass of water with which you can perform this trick?

[An alternative version of this trick, which you need not consider to receive full credit for this problem but may consider for fun, is to suck water up a thin straw, cap the top end of the straw with your finger, and behold the miracle of the water remaining suspended in the straw. This alternate trick is more complicated, however, because it is susceptible to effects due to the finite diameter of the straw (read: surface tension and the Rayleigh-Taylor instability, which you need not discuss, but may discuss if you would like)].

Atmospheric pressure keeps the card in place. That is, air molecules bombarding the bottom surface of the card impart enough upward momentum to keep it supported against the weight of the water. The tallest glass that this works for is about 33 feet (approximately 10 meters), derived as follows:

\[ F_{\text{up}} = P_{\text{atmosphere}} \times A \]

where \( F_{\text{up}} \) is the upward force of pressure on the bottom surface of the card, and \( A \) is the cross-sectional area of the card.

\[ F_{\text{down}} = m_{\text{water}} g = \rho V g = \rho A h g \]

where \( h \) is the height of the column of water and \( \rho \) is the density of water. To find the maximum height, set the two forces equal:

\[ \rho A h g = P_{\text{atm}} A \]
\[ \rho h g = P_{\text{atm}} \]
\[ P_{atm} = 101kPa = 1010 \times 10^3 \frac{dyne}{cm^2} \]

We know \( \rho = 1g/cm^3 \) and \( g = 980cm/s^2 \)

\[ h_{max} = \frac{P_{atm}}{\rho g} = \frac{1010 \times 10^3 \frac{dyne}{cm^2}}{(1 \frac{g}{cm^3})(980 \frac{cm}{s^2})} = 1031cm \approx 10m \]

The case of the straw with no card (which you did NOT have to talk about to receive full credit for this problem) is more complicated. This trick will not work for straws of arbitrarily large diameter. Yet in the calculation above, the diameter of the glass/straw is irrelevant. After personal experimentation, I have indeed found it impossible to get the trick to work for wide straws. I think the explanation here is that surface tension enables the trick to work in narrow straws. Atmospheric pressure is still responsible for supporting the column of liquid against its own weight. But surface tension stops the Rayleigh-Taylor instability from operating. The R-T instability rears its head every time dense fluid is accelerated into less dense fluid. Here gravity is accelerating the water into the air. The surface of the water breaks up into ripples; tongues of air slip into the water; air bubbles up to the top; and eventually the whole column of water evacuates. But surface tension prevents this from happening. I suspect that surface tension sets a minimum wavelength for the R-T ripples, and as long as the straw diameter is less than this wavelength, then the liquid is stable against R-T (we may study this in more detail later).

Note that surface tension alone is not responsible for keeping the liquid levitated. It’s still largely atmospheric pressure. Without atmospheric pressure, surface tension acting alone would levitate a measly column of liquid. Here’s how measly: take the weight of the water \( \pi r^2 h \rho g \) to be supported against surface tension, which supplies an upward force \( 2\pi \gamma r \), where \( \gamma = 70 \) dyne/cm is the surface tension of water (see Course Reader, page 1). For a straw of radius \( r \approx 2 \) mm, this gives a maximum height of \( h \approx \frac{2\gamma}{(\rho g)} \approx 0.7 \) cm, a far cry from my personal experience playing with soda straws on boring dates. I have tried to verify this estimate by removing my finger from the top end of the straw and seeing how much water remains in the straw. Indeed, it’s about a centimeter. I believe this is also known as “capillary action”.

Here’s still another puzzle related to the straw. This trick still seems to work even when one doesn’t suction the water first. That is, I can just insert the straw into a glass of water, cap the top of the straw with my finger, trapping in some air, and the water still levitates upon removing the straw from the glass. If the bottom of the straw experiences atmospheric pressure (acting up), and the top of the liquid in the straw ALSO feels atmospheric pressure from the trapped air (acting down), then the air pressure forces cancel and the liquid should evacuate the straw, contradicting observation. I think the
solution to this puzzle is that some liquid DOES evacuate the straw—leaving the trapped air at the top occupying a larger volume and therefore exerting less pressure. It doesn’t take much to make the air pressure at the top substantially less—just double the original volume of the trapped air, and you will have approximately halved the original pressure of the trapped air (assuming the trapped air behaves isothermally). Thus, a partial vacuum is created at the top, and pressure imbalance can be re-established.

Problem 2. Tip of the Iceberg

[10 points] An iceberg is sighted with volume $V$ above the water’s surface. What volume of ice must reside below the water’s surface? Take the density of berg ice, which is composed of frozen freshwater and full of bubbles, to be $\rho_{\text{ice}} = 0.9 \text{ g cm}^{-3}$, and the density of seawater to be $\rho_{\text{sea}} = 1.03 \text{ g cm}^{-3}$.

Idealize the iceberg as a rectangular slab (“ice cube”). The iceberg has area $A$, height $x$ above the water, and depth $y$ below the water. The given volume $V = Ax$.

The weight of the iceberg, $\rho_{\text{ice}} A (x + y) g$, exerts a pressure across its bottom surface of $\rho_{\text{ice}} (x + y) g$ (pressure is force divided by area). This pressure must be balanced by the water pressure. The water pressure at depth $y$ below the water’s surface is $\rho_{\text{sea}} g y$. So we have

$$\rho_{\text{ice}} (x + y) = \rho_{\text{sea}} y$$

which implies $y/(x + y) = \rho_{\text{ice}} / \rho_{\text{sea}} = 0.87$. But $y/(x + y)$ is just the fraction of the iceberg volume that is below the water. Therefore the volume of ice residing below the surface of the water is $1/(1-0.87) = \boxed{6.92}$ times greater than the volume of ice above.

Our idealization of a rectangular slab results in no loss of generality. For each directed area element $dA$ of the bottom of the iceberg, the force exerted upward against gravity by the water is proportional to $dA \cdot z$, which equals the flat-bottomed area element we considered above. (The force exerted by the water in the perpendicular direction is balanced against the intermolecular forces that resist compression.)

Problem 3. Scaling the Heights

[3 points for every part for a total of 18 points]

(a) Write down an expression for the variation of gas density $\rho$ (units of mass per volume) with height $z$ above the Earth’s surface, assuming the gas is ideal, at constant temperature $T$, and in hydrostatic equilibrium. Take the gas to be made of a single kind of molecule of weight $\mu$ (units of mass).\footnote{For this problem we take $\mu$ to have units of mass. In many other textbooks and papers, $\mu$ is dimensionless and gets multiplied by the mass of the hydrogen atom $m_{\text{H}}$; in that case, $\mu$ is called the “mean molecular weight” (which always struck me as a terrible name, because weight has dimensions). We will go back and forth between the conventions; there should be no confusion.} Express in terms of the density at ground level $\rho_0 \equiv \rho(z = 0)$ and the density scale height $h \equiv kT/(\mu g)$, where $g$ is the gravitational
acceleration. Neglect variations of $g$ with $z$.

Consider a parcel of gas with its bottom at height $z$, its top at $z + dz$, and horizontal surface area $A$. The gravitational force on the gas parcel is

$$F_g = -g \rho(z) A dz$$

There is also a force due to the pressure differential between the top and bottom of the parcel:

$$F_p = (P(z + dz) - P(z)) A$$

If the gas is ideal, then

$$P(z) = \frac{\rho(z) k T}{\mu}$$

Since there is no net force,

$$A \frac{k T}{\mu} (\rho(z + dz) - \rho(z)) = -g \rho(z) A dz$$

$$\frac{\rho(z + dz) - \rho(z)}{dz} = \frac{dp}{dz} = -\frac{\rho(z)}{h}$$

$$\frac{d\rho}{\rho} = -\frac{dz}{h}$$

Therefore,

$$\ln \left( \frac{\rho}{\rho_0} \right) = -\frac{z}{h}$$

So finally, we find that

$$\rho(z) = \rho_0 e^{-z/h}$$

(You could also have gotten the same result a bit more quickly by starting with hydrostatic equilibrium.)

(b) Neglecting collisions with other molecules, what is the maximum height a molecule would attain if launched from $z = 0$ with a typical thermal velocity? Is this close to $h$?

This calculation is highly misleading insofar as it ignores collisions between molecules, which are crucial for describing air as a continuum fluid. Nevertheless, it provides a mnemonic for remembering the scale height, and it illustrates the sometimes surprisingly close connection between fluid mechanics and particle mechanics (kinetic theory).

The translational kinetic energy of a molecule is $\frac{3}{2} k T$. By equating the initial kinetic energy to the final potential energy, we get that the maximum height achieved by this molecule will be

$$\frac{3}{2} k T = \mu g h_{max}$$

$$h_{max} = \frac{3 k T}{2 \mu g} = \frac{3}{2} h$$
So, $h_{max}$ is of order $h$.

(c) At what height $z$ would you expect the formula in (a), which depends on the continuum hypothesis, to fail? This height marks the location of the exobase in planetary atmospheres. Just consider how intermolecular collisions validate the continuum approximation.

The continuum hypothesis requires that particles collide a lot. Thus, we need:

$$h >> \lambda_{mfp}$$

Therefore, the hypothesis fails when $h \approx \lambda_{mfp}$. We know what $\lambda_{mfp}$ is:

$$\lambda_{mfp} = \frac{1}{n \sigma} = \frac{\mu}{\rho \sigma} = \frac{\mu \rho_0^{-1} e^{z/h} \sigma^{-1}}{\mu \rho_0^{-1} e^{z/h} \sigma^{-1}}$$

where $\sigma$ is some average cross section for air molecules.

So

$$\frac{h \rho_0 \sigma}{\mu} \approx e^{z/h}$$

$$z \approx h \ln \left( \frac{h \rho_0 \sigma}{\mu} \right)$$

(EC: The ln evaluates to something like 23. For $h = 8$ km, this means the exobase is about 190 km up.)

(d) Write down an expression for the hydrostatic variation of gas density $\rho$ with height $z$ above the midplane of a circumstellar disk at radius $r$. As in (a), assume constant $T$ and $\mu$. Take the gravitational field to be that from the star alone (ignore the self-gravity of the disk). Work in the limit that $z \ll r$. Express in terms of the density at the midplane $\rho_0$ and the density scale height $h \equiv (kT/\mu)^{1/2} \Omega^{-1}$, where $\Omega$ is the Keplerian orbital angular frequency.

The height $h$ is often written $h = c_s/\Omega$, where $c_s = (kT/\mu)^{1/2}$ is the speed of sound waves in gas that behaves isothermally (by behaving isothermally, we mean that the gas keeps the same temperature regardless of how it is compressed or expanded).

Once again we can start with hydrostatic equilibrium, again only considering the pressure in the $\hat{z}$ direction.

$$\frac{dP}{dz} = -\rho g_z$$

Considering only gravity from the star in the $\hat{z}$ direction,

$$g_z = \frac{GM}{r^2} \left( \frac{z}{r} \right)$$

in the limit $z \ll r$. We also have that the centripetal acceleration will be equal to the gravitational acceleration of the star

$$\Omega^2 r = \frac{GM}{r^2}$$
So $g_z = \Omega^2 z$. Then we have
\[ \frac{dP}{dz} = \frac{kT}{\mu} \frac{d\rho}{dz} = -\rho \Omega^2 z \]
\[ \frac{d\rho}{\rho} = -\frac{\mu \Omega^2}{kT} z \, dz \]

Let $h = (kT/\mu)^{1/2} \Omega^{-1}$. Then
\[ \frac{d\rho}{\rho} = -\frac{1}{h^2} z \, dz \]

Integrating from $z = 0$ to arbitrary $z$, we have
\[ \rho = \rho_0 e^{-\frac{z^2}{2h^2}} \]
which is a Gaussian.

(e) As in (b), calculate the maximum height that a gas molecule would attain if launched upwards from the midplane with a typical thermal speed, ignoring collisions. Compare to $h$.

Now potential energy will be given by
\[ U = \int_0^{h_{\text{max}}} \mu g_z \, dz = \int_0^{h_{\text{max}}} \mu \Omega^2 z \, dz = \frac{1}{2} \mu \Omega^2 h_{\text{max}}^2 \]

Equating this to the kinetic energy,
\[ \frac{1}{2} \mu \Omega^2 h_{\text{max}}^2 = \frac{3}{2} kT \]

\[ h_{\text{max}} = \left( \frac{3kT}{\mu \Omega^2} \right)^{1/2} = \sqrt{3} h \]

Again, we find that $h_{\text{max}}$ is of order $h$

(f) For the disk to be geometrically “thin” ($h/r \ll 1$), what must be true about $c_s/v_k$? Here $v_K$ is the Keplerian orbital velocity.

\[ \frac{c_s}{v_k} = \left( \frac{kT}{\mu} \right)^{1/2} \frac{1}{r \Omega} = \left( \frac{kT}{\mu \Omega^2} \right)^{1/2} \frac{1}{r^2} = \frac{h}{r} \]

So $h/r = c_s/v_k \ll 1$

**OPTIONAL Problem 4. To $d/dt$ or Not to $d/dt$**

[5 points] Give a physical explanation for why the left-hand side of the momentum equation reads $\rho du/dt$ and not $d(\rho u)/dt$, where $u$ is velocity, $\rho$ is density, and $d/dt$ is the convective derivative. What is being assumed about the fluid parcel in the Lagrangian view?
The momentum equation states, on its right-hand-side, all the forces acting to change the momentum of the fluid parcel as it travels in space. *These forces can change the volume of the fluid parcel but cannot change its mass.* (By changing its volume while preserving its mass, the fluid parcel can certainly change its density \( \rho \), contrary to what several of you stated. That is, we are *not* assuming incompressibility.)

The entire quantity \( \rho u \) is the momentum contained in *1 unit volume*. By taking \( d(\rho u)/dt \), we are asking what changes occur in 1 unit volume. But that presumes the parcel has 1 unit volume for all time as it travels in space. That is false.

On the other hand, \( u \) is the momentum contained in *1 unit mass*. By taking \( du/dt \), we are asking what changes occur in 1 unit mass. We are free to consider the parcel as having 1 unit mass (or a fixed number of unit masses). What is important is that the fluid parcel conserves this mass as it travels in space.

An alternative way of saying the above is given in Tritton, page 55.