Problem 1. The Many Guises of the Energy Equation

[10 points] Start from the total energy equation (Shu equation 2.32; we derived this in class):

\[
\frac{\partial}{\partial t} \left( \frac{\rho}{2} |u|^2 + \rho \varepsilon \right) + \frac{\partial}{\partial x_k} \left[ \rho \frac{1}{2} |u|^2 u_k + u_i (P \delta_{ik} - \tau_{ik}) + \rho \varepsilon u_k + F_k \right] = -\rho u_k \frac{\partial \phi}{\partial x_k} \tag{1}
\]

and derive the internal energy equation (Shu equation 2.36):

\[
\rho \frac{D \varepsilon}{Dt} = -P \nabla \cdot \vec{u} - \nabla \cdot \vec{F}_{\text{con}} + \tau_{ik} \frac{\partial u_i}{\partial x_k} \tag{2}
\]

The notation above follows that in lecture.

Problem 2. Torricelli’s Tank or, “A Hole in My Bucket”

A tall cylindrical tank having radius \( R_1 \) is filled with water. A nozzle of radius \( R_2 \ll R_1 \) is opened at the bottom of the tank, and water comes pouring out. Take the height of water in the tank to be \( h \) at some time.

(a) [5 points] Derive an expression for the velocity with which water emerges from the nozzle, assuming the flow is steady and inviscid.

(b) [5 points] The assumption of steady flow is not justified over arbitrarily long times, since the water level is continuously dropping. Estimate the timescale over which the flow can be approximated as steady.