Astrophysical Fluid Dynamics – Solution Set 2

By Anna Treaster, Jeff Silverman, Eugene Chiang, and Nicholas Rui

Readings: Shu pages 20–24, 30–33, 49–50; Tritton 10.4–10.7; Landau and Lifshitz Chapter 1, section 5, 6; Movies: e.g., Eulerian/Lagrangian, Fluid Quantity and Flow, Flow Visualization

Problem 1. The Many Guises of the Energy Equation

[10 points] Start from the total energy equation (Shu equation 2.32; we derived this in class):

\[
\frac{\partial}{\partial t} \left( \frac{\rho}{2} |u|^2 + \rho \varepsilon \right) + \frac{\partial}{\partial x_k} \left[ \frac{\rho}{2} |u|^2 u_k + u_i (P \delta_{ik} - \tau_{ik}) + \rho \varepsilon u_k + F_k \right] = -\rho u_k \frac{\partial \phi}{\partial x_k} \tag{1}
\]

and derive the internal energy equation (Shu equation 2.36):

\[
\frac{D \varepsilon}{Dt} = -P \nabla \cdot \vec{u} - \nabla \cdot \vec{F}_{\text{con}} + \tau_{ik} \frac{\partial u_i}{\partial x_k} \tag{2}
\]

The notation above follows that in lecture.

The momentum equation is:

\[
\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_k} (\rho u_i u_k + P \delta_{ik} - \tau_{ik}) = -\rho \frac{\partial \phi}{\partial x_i}
\]

Multiply the momentum equation by \( u_i \).

\[
u_i \frac{\partial}{\partial t} (\rho u_i) + u_i \frac{\partial}{\partial x_k} (\rho u_i u_k + P \delta_{ik} - \tau_{ik}) = -u_i \rho \frac{\partial \phi}{\partial x_i}
\]

Do some chain rule manipulations for the first two terms:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho |u|^2 \right) - \frac{1}{2} |u|^2 \frac{\partial \rho}{\partial t} - u_i \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k} \left( \frac{1}{2} \rho |u|^2 u_k \right) - \frac{\rho}{2} u_k u_i \frac{\partial u_i}{\partial x_k} - \frac{1}{2} |u|^2 \frac{\partial}{\partial x_k} (\rho u_k) = 0
\]

Now, using the continuity equation, \( \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} (\rho u_k) = 0 \), on the equation above, we get:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 \right) + \frac{\partial}{\partial x_k} \left( \frac{\rho}{2} |u|^2 u_k \right) = u_i \frac{\partial u_i}{\partial t} + \frac{\rho}{2} u_k u_i \frac{\partial u_i}{\partial x_k}
\]
So now plugging this result back into our full equation, we have

\[
\frac{\partial}{\partial t} \left( \frac{\rho}{2} |u|^2 \right) + \frac{\partial}{\partial x_k} \left( \frac{\rho}{2} |u|^2 u_k \right) + u_i \frac{\partial P}{\partial x_i} - u_i \frac{\partial \tau_{ik}}{\partial x_k} = -\rho u_i \frac{\partial \phi}{\partial x_i}
\]

subtract this equation from our original equation:

\[
\frac{\partial}{\partial t} (\rho \epsilon) + \frac{\partial}{\partial x_k} (u_i P \delta_{ik}) - \frac{\partial}{\partial x_k} (u_i \tau_{ik}) + \frac{\partial}{\partial x_k} (\rho \epsilon u_k) + \frac{\partial F_k}{\partial x_k} = u_i \frac{\partial P}{\partial x_k} - u_i \frac{\partial \tau_{ik}}{\partial x_k}
\]

expand out a couple of the derivatives:

\[
\frac{\partial}{\partial t} (\rho \epsilon) + P \frac{\partial u_k}{\partial x_k} + u_i \frac{\partial P}{\partial x_i} - u_i \frac{\partial \tau_{ik}}{\partial x_k} + \frac{\partial}{\partial x_k} (\rho \epsilon u_k) + \frac{\partial F_k}{\partial x_k} = u_i \frac{\partial P}{\partial x_k} - u_i \frac{\partial \tau_{ik}}{\partial x_k}
\]

After cancelling terms:

\[
\frac{\partial}{\partial t} (\rho \epsilon) + P \frac{\partial u_k}{\partial x_k} - \tau_{ik} \frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x_k} (\rho \epsilon u_k) + \frac{\partial F_k}{\partial x_k} = 0
\]

Expand via the product rule:

\[
\rho \frac{\partial \epsilon}{\partial t} + \rho \frac{\partial \rho}{\partial t} + \epsilon \frac{\partial}{\partial x_k} (\rho u_k) + \rho u_k \frac{\partial}{\partial x_k} (\epsilon) = -P \frac{\partial u_k}{\partial x_k} + \tau_{ik} \frac{\partial u_i}{\partial x_k} - \frac{\partial F_k}{\partial x_k}
\]

the second and third terms cancel because of the continuity equation.

Then converting the derivatives to vector notation, we get:

\[
\rho \frac{D \epsilon}{Dt} = -P \nabla \cdot \vec{u} - \nabla \cdot \vec{F} + \tau_{ik} \frac{\partial u_i}{\partial x_k}
\]

(3)

The end!

**Problem 2. Torricelli’s Tank or, “A Hole in My Bucket”**

A tall cylindrical tank having radius $R_1$ is filled with water. A nozzle of radius $R_2 \ll R_1$ is opened at the bottom of the tank, and water comes pouring out. Take the height of water in the tank to be $h$ at some time.
(a) [5 points] Derive an expression for the velocity with which water emerges from the
nozzle, assuming the flow is steady and inviscid.

According to the Bernoulli equation,

$$\int \frac{dP}{\rho} + \Phi + \frac{1}{2}v^2 = \text{constant}$$

Now for water, $\rho$ is to good approximation constant. So the enthalpy term is just $P/\rho$. Multiply the Bernoulli constant through by $\rho$:

$$P + \rho \Phi + \frac{1}{2} \rho v^2 = B$$

We evaluate the Bernoulli constant at two points in the flow. The first is at the top
of the tank. At the top, $P = P_{\text{atm}}$ (background atmospheric pressure). Also, provided
$R_2 \ll R_1$, the water level drops slowly, so we take $v = 0$. You can see this explicitly
from continuity:

$$v_{\text{tank}} \pi R_1^2 = v_{\text{nozzle}} \pi R_2^2$$

$$\frac{v_{\text{tank}}}{v_{\text{nozzle}}} = \left(\frac{R_2}{R_1}\right)^2 \ll 1$$

Finally, at the top, the water has a gravitational potential $\Phi = gh$. So at the top, we
have

$$P_{\text{atm}} + \rho gh = B_{\text{top}}$$

The second point of interest is at the bottom of the tank—and actually, just outside
the nozzle, where the water comes pouring out. At this exit point, $v = v$ (what we want
to solve for), $P = P_{\text{atm}}$, and $\Phi = 0$. So:

$$P_{\text{atm}} + \frac{1}{2} \rho v^2 = B_{\text{bottom}}$$

$$B_{\text{top}} = B_{\text{bottom}}$$ by the Bernoulli theorem. So:

$$\rho gh = \frac{1}{2} \rho v^2$$

$$v = \sqrt{2gh}$$
Incredibly enough, this is also the free-fall velocity!

(b) [5 points] The assumption of steady flow is not justified over arbitrarily long times, since the water level is continuously dropping. Estimate the timescale over which you expect your formula in (a) to be valid.

We will lose steady flow when \( h \) changes on the order of \( h_0 \), where \( h_0 \) is the original height in the tank (i.e. the time when \( \delta h \sim h \)).

\[
\frac{dV_{\text{water}}}{dt} = \pi R_1^2 \frac{\delta h}{\delta t}
\]

\[
\frac{dV_{\text{water}}}{dt} = \pi R_2^2 v(t)
\]

Obviously these two quantities are equal, barring evaporation and thirsty livestock.

\[
\pi R_1^2 \frac{\delta h}{\delta t} = \pi R_2^2 v(t)
\]

Now we impose our condition for losing steady flow: \( \delta h \sim h_0 \)

\[
R_1^2 \frac{h_0}{\delta t} = R_2^2 v(t)
\]

\[
\delta t = \left( \frac{R_1}{R_2} \right)^2 \frac{h_0}{v(t)}
\]

Then plugging in \( v(t) \) from part (a) we get:

\[
\delta t = \left( \frac{R_1}{R_2} \right)^2 \sqrt{\frac{h_0}{2g}}
\]