Astrophysical Fluid Dynamics – Problem Set 3

Readings: Tritton 5.8, chapter 2, chapter 8; Movies: Pressure Fields and Fluid Acceleration; whatever text you would like to read about the Taylor-Proudman Theorem (e.g., Acheson pages 279–280 works).

Problem 1. Running on All Cylinders

Consider a *barotropic* star—one whose pressure P depends only its density ρ. The star is self-gravitating and rotates about a central axis. The flow is steady, axisymmetric, and purely rotational (in the azimuthal direction only). Furthermore, the Rossby number of the flow is small; i.e., $u/(\Omega_0 L) \ll 1$, where the fluid velocity u is evaluated in a frame that rotates with the characteristic angular velocity of the star Ω_0, and L is a characteristic lengthscale of the flow. The Rossby number being small means that the star is nearly (but not exactly) rigidly rotating.

Prove that for such rotating barotropes, the angular velocity $\vec{\Omega} \equiv \Omega\hat{z}$ is constant on cylinders. That is, $\Omega = \Omega(r)$ is a function of cylindrical radius r only, measured from the rotation axis. (Picture the fluid to be a rotating oblate spheroid, and intersect the spheroid with an imaginary cylinder whose axis coincides with the spheroid's rotation axis. Then every fluid element on the surface of that cylinder rotates with the same angular velocity. Cylinders of different radii will correspond to different angular velocities [i.e., the fluid is not necessarily in rigid body rotation. We say the body is in *differential rotation* (on cylinders, in this case).]

Perhaps the most straightforward way to proceed is to take the curl of the momentum equation in a rotating frame. Be clear about your approximations.

This result is essentially a re-statement of the Taylor-Proudman theorem. However, most textbooks that give the T-P theorem assume that the *fluid* is incompressible. That assumption is not valid here. (Also, it is not necessary to understand the T-P theorem to do this problem.)

(Helioseismology tells us that in the outer convective shell of the Sun, the angular velocity is *not* constant on cylinders, but constant rather on surfaces of constant latitude. Explaining this observation remains an unsolved problem. By contrast, giant planet interiors are thought to be rotating more-or-less on cylinders.)

Problem 2. The Venturi Tube

The Venturi tube is a horizontal tube that is pinched (constricted) in the middle. Take the cross-sectional area of the unpinched part of the tube to be A_1 and the cross-sectional area of the pinched part of the tube to be A_2.

Atomic hydrogen gas flows steadily, inviscidly, and highly subsonically through the tube. Furthermore, the tube is insulated so that heat is neither lost nor gained from the tube to the outside world.

In the unpinched part of the flow, the gas density is measured to be ρ.

The pressure difference between the unpinched and pinched parts of the tube is measured to be ΔP.

Derive an expression for $\dot{m}$, the rate of mass flow through the tube [mass per time].

Problem 3. Reynolds Numbers

Estimate the Reynolds numbers characterizing

(a) water flow around a bluefin tuna

(b) air flow around a falling raindrop

(c) gas flows in a solar photospheric granule