Astrophysical Fluid Dynamics – Problem Set 4

Readings: Pringle & King 2.1.1 & 2.1.2; Tritton 5.8, chapter 2, chapter 8; Movies: Pressure Fields and Fluid Acceleration; whatever text you would like to read about the Taylor-Proudman Theorem (e.g., Acheson pages 279–280 works).

Problem 1. Dispersion Relation

[15 points for entire problem; point break-down specified below]

This problem derives in greater detail the properties of sound waves: waves whose restoring force is pressure, a.k.a. pressure waves. It utilizes a technique called Fourier mode analysis, a.k.a. linear mode analysis. The technique is applied ubiquitously in the literature to explore all manner of waves, and to uncover instabilities.

Consider (as we did in class) an infinite medium of constant pressure P_0 and constant density ρ_0 that is perfectly static $\vec{u}_0 = 0$. Perturb this medium in a *sinusoidal* manner:

$$P' \to P' \exp[i(\vec{k} \cdot \vec{r} - \omega t)] \tag{1}$$

$$\rho' \to \rho' \exp[i(\vec{k} \cdot \vec{r} - \omega t)] \tag{2}$$

$$\vec{u'} \to \vec{u'} \exp[i(\vec{k} \cdot \vec{r} - \omega t)] \tag{3}$$

where P', ρ', and $\vec{u'}$ on the right-hand-sides are constants representing the amplitudes—and relative phases—of the perturbations.[1] The perturbations are *linear* in the sense that $|P'| \ll P_0$ and $|\rho'| \ll \rho_0$; in other words, the perturbations are small. The wave frequency is ω, the wavevector is $\vec{k} = k\hat{k}$, $\hat{k}$ is the unit vector pointing in the direction of the wave, the wavenumber $k = 2\pi/\lambda$ is real, and λ is the wavelength of the disturbance. All of these are constants.

The position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

Comments:

[1] In writing these "arrow equations," it is understood that, e.g., P' on the right-hand-side (RHS) of the arrow is merely a constant coefficient (i.e., the amplitude), while P' on the left-hand-side is the full pressure field with its full time and space dependences. This is admittedly lazy notation, and if you're bothered by it, feel free to replace the arrow with an equal sign, and decorate the RHS primed quantities to distinguish them from their LHS counterparts — say with a dagger if you're feeling murderous. So e.g. $P' = P'^{\dagger} \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$. Keep the daggers on the amplitudes if you want; I'm going to skip them.

1. It is understood that when we write complex exponentials like this that we are really only interested in *either* the Real *or* the Imaginary parts.[2] Which part we choose is purely by convention. So for example, if we choose the convention that only the Real part is of interest, and IF P' is a real number (it need not be!—see point 2 below), then $P' \to P' \cos(\vec{k} \cdot \vec{r} - \omega t)$. This is merely a cosine disturbance in the background—a cosine across all of space at fixed time, and a cosine in time at fixed location.

   The point is that physical quantities like pressure and density are just real numbers, not complex ones (the weather forecast never says, "the pressure outside today is $1+3i$ bars"). The reason we wrap what is really a single real number into complex form is that (a) taking derivatives of exponentials is easy—even easier than taking derivatives of sines and cosines (in that we don't need to keep track of signs), and (b) the complex form enables us to keep track of *phase relationships* between different quantities, as we will see as part of this problem.

2. P', ρ', and $\vec{u'}$ (on the right-hand sides of equations 1–3) are constants (amplitudes)—but they can be, and are in general, *complex* constants, with real and imaginary components.

3. We have assumed the background is uniform, but really the analysis is good so long as the background doesn't change on scales comparable to or smaller than λ. That is, when we take spatial derivatives, the biggest spatial derivative will come from differentiating the wave $\exp(i\vec{k} \cdot \vec{r})$. This derivative is assumed to be much larger in magnitude than all other spatial derivatives.

4. There is no loss of generality in assuming sinusoidal (either cosine or sine) perturbations insofar as any perturbation can be Fourier decomposed into a sum of Fourier modes. What we are studying is the behavior of a single Fourier mode $(\omega, \vec{k})$; and the wonderful thing about linear theory is that every Fourier mode evolves independently of every other Fourier mode; i.e., linear waves obey *superposition.*

(a) [3 points] Use the continuity equation to relate ρ' to $\vec{u'}$, dropping all non-linear terms.

Simplify your final answer as much as possible (e.g., getting rid of bulky common terms like $\exp i(...)$).

Marvel at what you have done—you have converted a *partial differential equation* (the continuity equation) into an *algebraic equation.* Doing algebra is generally easier than solving PDEs! This is what Fourier analysis buys you.

(b) [3 points] Repeat as in (a), now using the momentum equation to relate P' to $\vec{u'}$.

(c) [3 points] Assume P is only a function of ρ (which it is in either the purely adiabatic or the purely isothermal cases for an ideal gas) to write $P' = (dP/d\rho)\rho'$, where $dP/d\rho$ is a known constant. Combine this with (a) and (b) to obtain:

[2] Using Euler's Formula $\exp i\theta = \cos\theta + i \sin\theta$—what Feynman called "the most remarkable formula in mathematics."

$$\omega^2 = (dP/d\rho)\vec{k}\cdot\vec{k} = (dP/d\rho)k^2 \tag{4}$$

Congratulations: you have derived the *dispersion relation* for sound waves. In general, the dispersion relation relates the wave frequency ω to the wavenumber $\vec{k}$. It is a *consistency* relation that must hold if the sinusoidal disturbances that we *declared* at the beginning to be true in equations (1)–(3) are physically realizable (i.e., obey the equations of mass, momentum, and energy).

In linear theory, dispersion relations should NOT depend on wave amplitudes. The dispersion relation governs all waves—as long as they are small in amplitude.

If for a given wavevector $\vec{k}$ the frequency ω is complex, then the disturbance can either decay (if $\mathrm{Im}(\omega) < 0$; see any of the defining equations 1–3) or grow (if $\mathrm{Im}(\omega) > 0$). The latter indicates *linear instability.* Are sound waves unstable?

(d) [1 point] Solve for the *phase velocity* $\vec{v}_{\rm phase} = (\omega/k)\hat{k}$. By definition, this is the speed at which a point of constant phase (say the crest of the wave) travels.

Waves are *dispersive* if the phase velocity varies with k. If dispersive, a given disturbance de-coheres as its constituent waves travel at different phase speeds; the disturbance "falls apart". Are sound waves dispersive?

(e) [2 points] Solve for the *group velocity* $\vec{v}_{\rm group} = \nabla_{\vec{k}}\omega \equiv \hat{x}\partial\omega/\partial k_x + \hat{y}\partial\omega/\partial k_y + \hat{z}\partial\omega/\partial k_z$. This is the velocity at which the envelope enclosing a wave packet (composed of waves of differing frequencies) travels (see your waves class).

(f) [2 points] Use (a)–(c) to decide the various phase relationships between $\vec{u'}$, P', and ρ'. When P' is reaching its maximum, what are $\vec{u'}$ and ρ' doing?

(g) [1 point] In what direction is the fluid velocity $\vec{u'}$ (NOT the wave velocities $\vec{v}_{\rm phase}$ and $\vec{v}_{\rm group}$)? Are sound waves *transverse* or *longitudinal*?

Problem 2. Running on All Cylinders

[20 points]

Consider a *barotropic* star—one whose pressure P depends only its density ρ. The star is self-gravitating and rotates about a central axis. The flow is steady, inviscid, axisymmetric, and purely rotational (in the azimuthal direction only). Furthermore, the Rossby number of the flow is small; i.e., $u/(\Omega_0 L) \ll 1$, where the fluid velocity u is evaluated in a frame that rotates with the mean angular velocity of the star Ω_0, and L is a characteristic lengthscale of the flow. The Rossby number being small means that the star is nearly (but not exactly) rigidly rotating.

Prove that for such rotating barotropes, the angular velocity $\vec{\Omega} \equiv \Omega\hat{z}$ is constant on

cylinders. That is, $\Omega = \Omega(r)$ is a function of cylindrical radius r only, measured from the rotation axis, and does not depend on vertical height z.[3]

Perhaps the most straightforward way to proceed is to take the curl of the momentum equation in a frame rotating at the mean angular velocity $\vec{\Omega}_0 = \Omega_0 \hat{z}$. Be clear about your approximations, and remember that in the rotating frame there are "fictitious" accelerations (hint: there are 2, and you should keep both).

This result is essentially a re-statement of the Taylor-Proudman theorem. However, most textbooks that give the T-P theorem assume that the *fluid* is incompressible. That assumption is not necessary, so do not use it. (Also, it is not necessary to understand the T-P theorem to do this problem.)

(Helioseismology tells us that in the outer convective shell of the Sun, the angular velocity is *not* constant on cylinders, but constant rather on surfaces of constant latitude. Similarly, Jupiter and Saturn almost certainly do not respect purely cylindrical flow. Identifying the forces that break the Taylor-Proudman state is a forefront problem.)

Useful math facts (google "vector identities"; see also beginning pages of the Course Reader):

- The curl of a gradient is zero.
- For any two vector fields $\vec{a}$ and $\vec{b}$, $\nabla \times (\vec{a} \times \vec{b}) = \vec{a}(\nabla \cdot \vec{b}) - \vec{b}(\nabla \cdot \vec{a}) + (\vec{b} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{b}$
- For any scalar field a and vector field $\vec{b}$, $\nabla \times (a\vec{b}) = a(\nabla \times \vec{b}) + \nabla a \times \vec{b}$

[3] To help visualize this, picture the fluid to be a rotating oblate spheroid, and intersect the spheroid with an imaginary cylinder whose axis coincides with the spheroid's rotation axis. Then every fluid element on the surface of that cylinder (every fluid element on the cylinder "wall") rotates with the same angular velocity. Cylinders of different radii will correspond to different angular velocities, that is, the fluid is not necessarily in rigid body rotation. We say the body is in *differential rotation* (on cylinders, in this case).