Astrophysical Fluid Dynamics – Problem Set 4

Readings: Tritton, chapter 3; Movies: Drag I and II; also readings from textbooks by Acheson and by Batchelor as pointed out in specific problems below

Problem 1. Elementary Viscous Flows

Problems taken from Acheson (Chapter 2; Problem 2.3; Problem 7.2). These are "classic problems" in low Reynolds number (highly viscous) flows. While they are not directly germane to astrophysics, they do provide practice with solving the Navier-Stokes equation (highly simplified), and they arguably form part of the standard canon of fluid mechanics which physicists of all persuasions should know.

Consider an incompressible fluid of dynamic viscosity μ. For all problems below, take the Reynolds number to be $\ll 1$ and neglect gravity.

(a) *Steady flow between plates*: Viscous fluid flows between two stationary rigid boundaries $y = \pm h$ under a constant pressure gradient $G = -dP/dx$. Show that

$$u = \frac{G}{2\mu}(h^2 - y^2), \quad v = w = 0\,. \tag{1}$$

(b) *Poiseuille flow*, after the physician who studied such flows in connection with blood vessels: Viscous fluid flows steadily down a horizontal pipe of circular cross-section $r = a$ under a constant pressure gradient $G = -dp/dx$. Show that

$$u = \frac{G}{4\mu}(a^2 - r^2) \tag{2}$$

(Such pipe flow is unstable at high Re, as discovered by Osborne Reynolds, the father of the study of fluid turbulence.)

(c) *Couette flow*: Viscous fluid occupies the space between two infinitely long, concentric cylinders of radius r_1 and $r_2 > r_1$. The cylinders rotate at angular speeds of Ω_1 and Ω_2, respectively. Show that in steady state, the fluid rotates with velocity

$$u = u_\theta = Ar + \frac{B}{r} \tag{3}$$

where $A = (\Omega_2 r_2^2 - \Omega_1 r_1^2)/(r_2^2 - r_1^2)$ and $B = (\Omega_1 - \Omega_2)r_1^2 r_2^2/(r_2^2 - r_1^2)$.

(Such flow becomes unstable to the formation of Taylor vortices if $\Omega_1 \gg \Omega_2$. Maybe we'll study this instability later.)

(d) Imagine a rigid sphere of radius a immersed in the viscous fluid, initially rotating at angular frequency Ω. Write down an order-of-magnitude expression for the spin-down time of the sphere. For extra credit: evaluate this spin-down time for the Earth's inner solid iron core, immersed in the outer liquid iron core.

Problem 2. Cavitation

Problem taken from Blandford and Thorne, Exercises 12.11–12.12

A submarine lies at depth $h \approx 500\,\mathrm{m}$ below the water's surface.

(a) The propeller radius is $L \approx 2\,\mathrm{m}$. Estimate the angular rotation speed Ω for which cavitation—the creation of bubbles of water vapor near the fast-moving propeller—occurs. Provide a symbolic answer and a numerical answer in revolutions per second.

(b) The bubble, created with radius R_i, travels to a location where the ambient pressure is P_0, much greater than the pressure where the bubble was created. The greater pressure causes the bubble to collapse. The rest of this problem studies this implosion.

We picture the water surrounding the bubble to undergo spherical infall with velocity $v(r,t)$, where r measures the radial distance from center of the bubble. Show that as a result of water's incompressibility ($\rho =$ constant),

$$v = \frac{F(t)}{r^2} \tag{4}$$

where $F(t)$ is a function solely of time (and not space).

(c) By integrating the momentum equation from the bubble's surface at $r = R$ to $r = \infty$, show that the fluid velocity $v(R)$ at the bubble's surface obeys

$$-\frac{1}{R}\frac{dF}{dt} + \frac{1}{2}v(R)^2 = \frac{P_0}{\rho} \tag{5}$$

(d) Show that

$$|v(R)| = \left(\frac{2P_0}{3\rho}\right)^{1/2} \left[\left(\frac{R_i}{R}\right)^3 - 1\right]^{1/2} \tag{6}$$

(e) The yield stress of a submarine hull might be $Y \sim 10^9\,\mathrm{dyne\,cm^{-2}}$. Estimate the factor by which the bubble shrinks, R_i/R, such that the imploding bubble would be expected to damage the hull.

Problem 3. Turning the Bend

Consider a short cylindrical vat of radius R containing incompressible, viscous fluid of density ρ and kinematic viscosity ν. The vat (not the fluid) is set spinning at angular velocity Ω. The Reynolds number Re_R is very small. Eventually the liquid comes into solid body rotation with the vat (via the Ekman boundary layer, which we might study in the future.)

(a) Calculate the run of pressure P with radius r, after solid body rotation sets in. Neglect atmospheric pressure and neglect also hydrostatic pressure due to gravity (the vat is very short). Take the pressure at the center of the vat to be $P_0 = P(r = 0)$.

(b) Place the long side of your index finger underneath a gently running tap so that water falls only on one side of your finger. Observe how the stream tends to hug the underside of your finger and get deflected towards the other (unwetted) side.

You can also try this with a spoon. Turn the spoon sideways and place the bottom of the spoon into the flow so that water curves round the spoon. By moving the spoon in and out of the flow, feel how the spoon seems to be *attracted* into the running water. (It should feel like a kitchen magnet placed very close to the fridge.)

Based on your understanding of (a), derive an order-of-magnitude expression for the horizontal force exerted on your finger, per unit length of wetted finger. Express your answer in whatever variables you think are important.

The effect that this problem explores is called the *Coanda* effect, and is reported to be the same effect responsible for liquids refusing to pour nicely out of containers. If you pour beer from a pitcher that is tipped at too small an angle from the vertical, the beer leaves the rim only to dribble down the side of the pitcher.

OPTIONAL Problem 4. Symmetry of Stress Tensor

We introduced the stress tensor σ_{ij} as a "box of 9 numbers" describing the force in the j-direction exerted on a surface of constant i. This problem shows that the 9 numbers actually aren't all independent; in fact the stress tensor is symmetric: $\sigma_{ij} = \sigma_{ji}$, so that there really are only 6 independent numbers. This result was shown by Cauchy. The following problem is taken from textbooks by Acheson (Chapter 6; Problem 6.14) and by Batchelor (page 11). It isn't that hard; it's essentially an exercise in tensor calculus, and it proves a result that isn't that obvious.[1]

[1] It wasn't obvious, I think, to Newton, whose hypothesis for the viscous stress tensor $\tau_{ij} = \mu \partial u_j / \partial x_i$ is incorrectly asymmetric. By contrast, an example of a symmetric stress tensor is $\tau_{ij} = \mu(\partial u_j/\partial x_i + \partial u_i/\partial x_j)$. Fluids that obey this relation are called Newtonian—giving generous credit to Sir Issac! This symmetric form and Newton's asymmetric form share one key property, however, which is that the stress is linearly proportional to the velocity gradient.

Reynolds' transport theorem states that

$$\frac{d}{dt}\int_{V(t)} G dV = \int_{V(t)} \left(\frac{dG}{dt} + G\nabla\cdot\mathbf{u}\right) dV \tag{7}$$

where $V(t)$ denotes the region of space occupied by a deforming fluid parcel, $\mathbf{u}$ is the velocity, $G(\mathbf{x},t)$ is any scalar or vector function, and d/dt is the convective derivative.

(a) Argue[2] that the theorem makes sense by considering $G = \rho F$, where $\rho(\mathbf{x},t)$ is the fluid density and $F(\mathbf{x},t)$ is any function. That is, show that the above statement implies

$$\frac{d}{dt}\int_{V(t)} F\rho dV = \int_{V(t)} \frac{dF}{dt}\rho dV \tag{8}$$

and explain why this latter statement makes sense. (Hint: what is conserved about the fluid parcel?)

(b) Conservation of angular momentum of the fluid parcel states that

$$\frac{d}{dt}\int_{V(t)} \mathbf{x}\times\rho\mathbf{u} dV = \int_{S(t)} \mathbf{x}\times\mathbf{t} dS + \int_{V(t)} \mathbf{x}\times\rho\mathbf{f}_{\text{body}} dV \tag{9}$$

where $t_i dS = \sigma_{ij} n_j dS$ is the force in the i-direction acting on a surface having a directed area element $\hat{\mathbf{n}}dS$, $S(t)$ denotes the surface of the fluid parcel, and $\mathbf{f}_{\text{body}}$ contains all the body forces. The symbol $\times$ denotes the cross product: e.g., $(\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}a_j b_k$, where ϵ_{ijk} is the (space- and time-independent) Levi-Civita tensor. Clearly, the first term on the right-hand side contains the torques due to surface stresses, while the second term contains the torques due to body forces.

Use Reynolds' transport theorem and our standard momentum equation to show that

$$\int_{V(t)} \epsilon_{ijk} x_j \frac{\partial\sigma_{kl}}{\partial x_l} dV = \int_{S(t)} \epsilon_{ijk} x_j t_k dS \tag{10}$$

(c) Use the divergence theorem to rewrite the surface integral above as

$$\int_{S(t)} \epsilon_{ijk} \frac{\partial(x_j\sigma_{kl})}{\partial x_l} dV\,. \tag{11}$$

Thereby show that

[2] A complete proof is outlined in problem 6.13 of Acheson; the solution is in the back of the book.

$$\int_{V(t)} \epsilon_{ijk}\sigma_{kj} dV = 0 \tag{12}$$

and that $\sigma_{kj} = \sigma_{jk}$ as a consequence.