

Astrophysical Fluid Dynamics – Problem Set 5

Readings: Shu pages 73–81; Course Reader selections from Frank, King, & Raine

Problem 1. There Can Be Only One (Not Counting Sequels)

Consider spherically symmetric in/out-flow near an object of mass M . Assume the flow is isothermal with sound speed c_s . Define the Mach number $\mathcal{M} = u/c_s$ for radial velocity u .

(a) [5 points] Show that

$$\left(\frac{1}{\mathcal{M}} - \mathcal{M}\right) \frac{\partial \mathcal{M}}{\partial r} = \frac{GM}{c_s^2 r^2} - \frac{2}{r} \quad (1)$$

where r measures radial distance from the object and G is the gravitational constant.

(b) [10 points] Show that

$$\frac{\mathcal{M}^2}{2} - \ln \mathcal{M} = 2 \frac{r_s}{r} + 2 \ln \frac{r}{r_s} + C \quad (2)$$

where $r_s \equiv GM/2c_s^2$ gives the radius of the “sonic point” and C is a constant of integration.

Plot several solutions of \mathcal{M} versus r , drawing from all mathematically possible families. Label the solutions by their respective C values. Identify those subset of solutions that are physical.

Problem 2. The Solar Wind

At the base of the solar corona at $r \approx R_\odot$, outward flow speeds are still subsonic. Spectral line observations of the coronal base reveal temperatures of about $T \approx 2 \times 10^6$ K, a fully ionized plasma, and a number density of protons of about 10^8 cm^{-3} .

You should use the results of Problem 1.

(a) [5 points] Estimate \dot{M} for the solar wind, assuming the wind is isothermal. Express in $M_\odot \text{ yr}^{-1}$.

(b) [5 points] Estimate the number density of solar wind protons flying past the Earth. Express in cm^{-3} . You should NOT have to assume the wind is moving at constant velocity past the sonic point.

Problem 3. Jet Collimation by Pressure Confinement from an External Medium

Adapted from Pringle and King, problem 3.4. The problem starts by considering a de Laval nozzle, and ends by considering how astrophysical *jets* might behave like such nozzles—with the tube of given cross-sectional area replaced by an external medium of a given pressure profile.

Consider a γ -law gas ($P \propto \rho^\gamma$) moving inviscidly through a de Laval nozzle. The gas initially is very subsonic, but eventually moves supersonically. Take the sound speed of the gas initially to be c_{s0} .

(a) [2 points] Solve for the gas velocity u and the sound speed c_s at the pinched throat of the nozzle, in terms of c_{s0} and γ .

(b) [1 point] Solve for the gas velocity u at infinity, in terms of c_{s0} and γ .

(c) [2 points] Solve for the pressure P at the pinched throat, in terms of the initial P_0 and γ . Does the pressure decrease monotonically everywhere along the tube? Increase monotonically? Neither? Justify your answer.

(d) [5 points] Now consider an astrophysical jet, and how it is collimated. One proposed mechanism for collimation is pressure confinement by an external, static medium through which the jet travels. For example, the jet could be a stellar jet (powered by accretion from a disk), and the external medium could be the remnant gas cloud from which the star formed. Or the jet could be an AGN (supermassive black hole) jet (powered by accretion from a disk), and the external medium could be gas in the core of the galaxy.

There are two ideas here. The first is that the external, static medium has a thermal pressure that decreases monotonically in the jet direction. The second is that the jet and the external medium reach pressure equilibrium. That is, the thermal pressure at any point within the jet equals the local thermal pressure of the external surrounding medium. Once in pressure equilibrium, the jet stops expanding in the transverse direction (some papers refer to the jet as being “cocooned.”) These two ingredients, coupled with the usual assumption of steady inviscid flow, imply a close analogy between de Laval nozzles and jets. Instead of being given the tube’s cross-sectional area $A(x)$ and solving for $P(x)$, we are given $P(x)$ and solve for the jet cross-sectional area $A(x)$.

Consider a steady jet composed of a gas having $\gamma = 7/5$. Far along the jet, we are so far from the “central engine” (i.e., whatever is driving the jet) that we can neglect gravity. Moreover, the jet at this stage has achieved a constant supersonic velocity (analogous to part b of this problem). The jet burrows into an external medium whose pressure scales with distance along the jet as x^{-2} (like that of a singular isothermal sphere; or an isothermal wind).

Under these conditions, how does the *opening angle* of the jet (transverse length divided by longitudinal length x) scale with x ? Take the jet to have a circular cross-section, and justify whatever assumptions you make.

(The idea of jet as de Laval nozzle was introduced by [Blandford & Rees 1974](#) in the context of relativistic AGN jets (some pages are reprinted in the Course Reader). There also exists a Bernoulli constant for relativistic flows, but it looks different from our non-relativistic form $u^2/2 + \int dP/\rho + \Phi$. After thinking about this problem, you should find at least the first few pages of their paper to be readable; you will also appreciate their Figure 1.)