Problem 1. *LOST*

Desmond turns the key and your plane splits in half at an altitude of 32,000 feet above the Pacific. You are not wearing your seatbelt and go flying out.

(a) About how long does it take for you to hit the water’s surface? Credit awarded for detailed physical arguments.

The two important forces acting on me after I leave the plane are gravity and drag. To figure out the drag force let’s first determine my Reynolds number as I’m falling through the air. The kinematic viscosity of air is $\nu_{\text{air}} \sim 2 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1}$. The relevant length scale is of the order $L \sim 1 \text{ m}$, and I expect my velocity will be on the order of $u \sim 10 \text{ m s}^{-1}$. This implies that $Re \sim uL/\nu \sim 10^6 \gg 1$. We are in the high Reynolds number limit: inertial forces will dominate over viscous forces, and the drag force is given by

$$F_D = \frac{1}{2} C_D \rho A v^2$$

$$m \frac{dv}{dt} = Fg + F_D = mg - \frac{1}{2} C_D \rho A v^2$$

$$\frac{dv}{dt} = g - \frac{1}{2} \frac{C_D \rho A v^2}{m}$$

To find the time it takes me to fall a distance $d = 32,000 \text{ ft}$, I will approximate $t_{\text{fall}} \approx d/v_T$, where $v_T$ is the terminal velocity. $v_T$ is given by

$$\frac{dv}{dt} = 0 \Rightarrow mg = \frac{1}{2} C_D \rho A v_T^2$$

$$v_T = \sqrt{\frac{mg}{\frac{1}{2} C_D \rho A}}$$

I am going to try to keep my body horizontal to maximize drag. White gives the drag of an average horizontal person at $Re \geq 10^4$ to be $C_D A \sim 9 \text{ ft}^2 \sim 0.8 \text{ m}^2$. I’ll use $m = 60 \text{ kg}$. For $\rho$ I’ll use the average density of air over my fall:

$$\rho = < \rho > = \frac{\rho_{\text{air, sea level}} + \rho_{\text{air, 32000 ft}}}{2}$$
The density of air at sea level is \(0.0012 \text{ g cm}^{-3}\). As we showed in problem set 1, \(\rho(z) = \rho_0 e^{-z/h}\). Assuming the atmosphere is made predominately of nitrogen and has a temperature of 300 K, we find that the scale height \(h = \frac{kT}{\mu g} \sim 46,000 \text{ ft}\). Therefore we find that

\[
\rho_{\text{air,32000 ft}} = 0.0012 \text{ g cm}^{-3} e^{-\frac{32000}{46000}} = 0.0006 \text{ g cm}^{-3} \Rightarrow <\rho> = 0.0009 \text{ g cm}^{-3}
\]

Now we’re ready to plug in for \(v_T\):

\[
v_T = \sqrt{\frac{mg}{\frac{1}{2}C_D \rho A}} = \sqrt{\frac{2(60 \text{ kg})(1000 \text{ cm s}^{-2})}{(0.8 \text{ m}^2)(0.0009 \text{ g cm}^{-3})}} = 41 \text{ m s}^{-1}
\]

This gives a falling time of

\[
t_{\text{fall}} \approx \frac{d}{v_T} = 32,000 \text{ ft}/41 \text{ m s}^{-1} = 4 \text{ minutes}
\]

This is the answer, as long as we can neglect the fact that we aren’t always traveling at \(v_T\). To check this, let’s see how long it takes to reach terminal velocity, neglecting drag:

\[
\frac{dv}{dt} = g
\]

\[
v_T \sim g \Delta t \Rightarrow \Delta t \sim \frac{v_T}{g} \sim \frac{41 \text{ m s}^{-1}}{1000 \text{ cm s}^{-2}} \sim 4.1 \text{ s}
\]

Since \(\Delta t \ll t_{\text{fall}}\) we can neglect the effects of this initial acceleration. So our final answer remains

\[
t_{\text{fall}} = 4 \text{ minutes}
\]

(b) About how deep do you penetrate the water? Credit awarded for detailed physical arguments.

The force equation is

\[
m \frac{dv}{dt} = F_g - F_B - F_D
\]

where \(F_B\) is the buoyant force. When I am fully submerged, \(F_B \approx F g\) (because \(\rho_{\text{person}} \approx \rho_{\text{water}}\)). Plus, at high velocities \(F_D\) will dominate. So I’ll estimate that

\[
m \frac{dv}{dt} = -F_D = -\frac{1}{2} C_D \rho A v^2
\]

Let \(C = \frac{C_D \rho A}{2m}\):

\[
\frac{dv}{dt} = -C v^2
\]

\[
\int_{v_f}^{v} \frac{dv}{v^2} = -\int_{0}^{t(v)} C dt'
\]
\[-\frac{1}{v} + \frac{1}{v_T} = -Ct\]

\[v(t) = \frac{v_T}{1 + Cv_T t}\]  \hfill (1)

Integrating again:

\[
\int_0^{x(t)} dx = -\int_0^t dt' \frac{v_T}{1 + Cv_T t'}
\]

\[x(t) = \frac{1}{C} \ln(1 + Cv_T t)\]  \hfill (2)

Rearranging (1), we have

\[t(v) = \frac{1}{Cv_T} \left( \frac{v_T}{v} - 1 \right)\]

Plugging this into (2):

\[
\boxed{x(v) = \frac{1}{C} \ln \left( \frac{v_T}{v} \right)}
\]

Unless I swim I am never going to stop, so I’ll find the depth at which I slow to .01 \( v_T = 0.44 \) m s\(^{-1}\)

\[
\Rightarrow x(\cdot01 \ v_T) = \frac{2m}{C_D p A} \ln(100)
\]

As in (a) I’ll use \( m = 60 \) kg. The density of water is 1 g cm\(^{-3}\). I am hoping to go in feet first, so I don’t break my back. So I’ll use \( C_D A = 0.1 \) m\(^2\), as given by White for an average vertical person at \( Re \geq 10^4 \). Plugging in I find

\[
\boxed{x(\cdot01 \ v_T) = 5.5 \text{ m}}
\]

If I enter the water at terminal velocity, I expect to be submerged to a depth of \( \sim 5.5 \) m.

**Problem 2. Particle Drift in Protoplanetary Disks**

Planets form in disks of gas and dust surrounding young stars.

Consider the young Sun, encircled by a disk of gas of surface density (mass per unit face-on area) \( \Sigma \approx 10^3 \) g cm\(^{-2}\) in the vicinity of 1 AU. The temperature of disk gas is \( T \approx T_0(a/\text{AU})^{-q} \), where \( T_0 \approx 200\) K, \( a \) is the disk radius, and \( q \approx 1/2 \). Call the corresponding sound speed \( c_s \approx \sqrt{kT/\mu} \), where \( k \) is Boltzmann’s constant and \( \mu \) is the mean molecular weight.

(a) Consider a dust particle of size \( s = 1 \) \( \mu \)m and internal density \( \rho_p \approx 1 \) g cm\(^{-3}\) located one scale height above the disk midplane, \( z = h \) (see problem set 1). Estimate the time the particle takes to settle vertically to the midplane. Provide both a symbolic answer and a numerical answer in years.

Once the dust particle reaches terminal velocity the force equation is

\[F_g \ddot{z} - F_D = 0\]
We wrote down \( F_g \dot{z} \) for material in a circumstellar disk in Problem Set 1. It is given by:

\[
F_g \dot{z} = m\Omega^2 z = \frac{4\pi \rho_p s^3}{3} \Omega^2 h
\]

At one scale height \( h = \frac{c_s}{\Omega} \), so

\[
F_g \dot{z} = \frac{4\pi \rho_p s^3}{3} \Omega c_s
\]

Now, what is \( F_D \)? To figure out what regime we are in, we must compare \( s \) to \( \lambda_{mfp} \). To estimate \( \lambda_{mfp} \) I assume the gas in the disk is made of molecular hydrogen with a gas density of \( \rho_{gas} = 10^{-9} \text{ g cm}^{-3} \) (from Imke de Pater’s book). This gives

\[
n = \frac{\rho_{gas}}{2m_p} \approx 3 \times 10^4 \text{ cm}^{-3}
\]

Using our usual estimate \( \sigma \approx \left(\frac{3\AA}{\sigma}\right)^2 \approx 10^{-15} \text{ cm}^2 \) we find that

\[
\lambda_{mfp} = \frac{1}{n\sigma} \approx \left(3 \times 10^4 \text{ cm}^{-3} \times 10^{-15} \text{ cm}^2\right)^{-1} \approx 3 \text{ cm}
\]

Since the grain size \( s = 1 \mu\text{m} \) we see that \( \lambda_{mfp} > s \). The other thing we need to know is whether we are in the subsonic or supersonic regime. Let’s assume we are subsonic and check later. In that case, we are in the limit of Epstein drag:

\[
F_D \approx 4 \rho c_s v \pi s^2
\]

In this case \( \rho \) is \( \rho_{gas} \) which we can write in terms of surface density \( \Sigma \):

\[
\Sigma = \int_{-\infty}^{\infty} \rho_{gas} dz \approx 2\rho h = \frac{2\rho_{gas} c_s}{\Omega}
\]

\[
\rho_{gas} = \frac{\Omega \Sigma}{2c_s} ; \quad F_D \approx 2\Omega \Sigma v \pi s^2
\]

Plugging into \( F_g \dot{z} = F_D \),

\[
\frac{4\pi \rho_p s^3 \Omega c_s}{3} = 2\Omega \Sigma v \pi s^2
\]

\[
v = \frac{2\rho_p s c_s}{3\Sigma}
\]

At this point we can check our subsonic assumption

\[
\frac{v}{c_s} = \frac{2\rho_p s}{3\Sigma} \approx \frac{10^{-4}}{10^4} \ll 1
\]

so the subsonic assumption was good. Now, the particle will never actually reach the plane, so I’ll estimate the timescale over which the particle falls half the distance to the plane:

\[
t_{1/2} = \frac{h/2}{v} = \frac{c_s}{2\Omega v}
\]

4
\[ t_{1/2} = \frac{3\Sigma}{4\rho_p\Omega} \]

At 1 AU, \( \Omega = \frac{2\pi}{1\text{yr}} \). For \( M = M_{\odot} \)

\[ t_{1/2} = \frac{3(10^3)}{8\pi(1)(10^{-4})} \text{ years} = 1.2 \times 10^6 \text{ years} \]

or about 1 Myr.

(b) Does disk gas rotate at sub-Keplerian or super-Keplerian speeds? Estimate the difference \( \Delta v \) between the gas velocity and the circular Keplerian speed \( v_K \equiv \Omega a \equiv \sqrt{GM_\odot/a} \), at the midplane \( z = 0 \) at \( a = 1 \text{ AU} \). Express \( \Delta v \) symbolically in terms of \( c_s, \Omega, a \), and whatever dimensionless numbers you need. Also provide a numerical estimate.

The gas in the protoplanetary disk will be partially supported by the outward radial pressure gradient; therefore the gas speed will be Sub-Keplerian. Starting again with a force equation:

\[ a\Omega^2 = \frac{v^2}{a} = \frac{GM}{a^2} + \frac{1}{\rho} \frac{dP}{da} \]

\[ P = \rho c_s^2 \Rightarrow \frac{1}{\rho} \frac{dP}{da} = \frac{-c_s^2}{a} \]

\[ \frac{v^2}{a} = \frac{GM}{a} - \frac{c_s^2}{a} \]

\[ v = \sqrt{\frac{GM}{a} - \frac{c_s^2}{a}} \]

Using \( v_k = \sqrt{\frac{GM}{a}} \),

\[ v = \sqrt{v_k^2 - \frac{c_s^2}{a}} = v_k \sqrt{1 - \frac{c_s^2}{v_k^2}} \]

\[ \Delta v = v_k - v = v_k \left(1 - \sqrt{1 - \frac{c_s^2}{v_k^2}}\right) \approx v_k \left(\frac{c_s^2}{v_k^2}\right) \]
Pluggin in for $v_k$, we get $\Delta v \approx \frac{c_s^2}{\Omega a}$. At $a = 1$ AU, $\Omega = \frac{2\pi}{1 \text{ yr}}$, and $M = M_\odot$, and estimating $c_s^2 = \frac{kT}{\mu} = \frac{k(200K)}{2m_p}$, we get:

$$\Delta v = 28 \text{ m s}^{-1}$$

This is how much slower the gas is moving relative to the Keplerian speed.

(c) Particles (read: rocks) are considered well-entrained in disk gas—i.e., they rotate with nearly the same velocity as disk gas—if their momentum stopping time

$$t_{\text{stop}} \equiv \frac{mv_{\text{rel}}}{F_D}$$

is short compared to the dynamical time $t_{\text{dyn}} \sim 1/\Omega$ [this latter time is the time for disk gas to turn / change its velocity (direction) by $\sim 1$ radian]. Here $m$ is the particle mass, $v_{\text{rel}}$ is the relative velocity between a particle and disk gas, and $F_D$ is the drag force on the particle.

Estimate the critical size $s_{\text{crit}}$ for which $t_{\text{stop}} \sim t_{\text{dyn}}$ at the midplane $z = 0$ at $a = 1$ AU. A numerical answer suffices. Are bodies larger or smaller than $s_{\text{crit}}$ well-entrained?

We want to solve for when $t_{\text{stop}} \sim t_{\text{dyn}}$, e.g. when

$$\frac{m\Delta v}{F_D} \sim \frac{1}{\Omega}$$

Plugging in for Epstein drag, we can solve for $s_{\text{crit}}$: 

$$\frac{m\Delta v}{4\rho_c \Delta v \pi s_{\text{crit}}^2} \sim \frac{1}{\Omega}$$

$$s_{\text{crit}}^2 \sim \frac{m\Omega}{4\pi \rho_c s_{\text{crit}}}$$

$$\sim \frac{2\pi \Sigma}{m}$$

$$\sim \frac{1}{2\pi \Sigma} \frac{4\pi \rho_p s_{\text{crit}}^3}{3}$$

$$\sim \frac{2\rho_p s_{\text{crit}}^3}{3\Sigma}$$

$$s_{\text{crit}} \sim \frac{3\Sigma}{2\rho_p}$$

$$\sim \frac{3(10^3 \text{ g cm}^{-2})}{2(1 \text{ g cm}^{-3})}$$

$$\sim 15 \text{ m}$$
But we have a problem: \( s_{\text{crit}} > \lambda_{mfp} \Rightarrow \) we aren’t in the Epstein drag limit! So let’s repeat the equation using the correct drag equation, \( F_D = \frac{1}{2} C_D \rho A \Delta v^2 \):

\[
\frac{1}{\Omega} \sim \frac{m \Delta v}{\frac{1}{2} C_D \rho A \Delta v^2} \]

\[
1 \sim 2\Omega \frac{4 \pi \rho_p s_{\text{crit}}^3}{C_D \rho \text{gas} \pi s_{\text{crit}}^2 \Delta v} \sim 8\Omega \rho_p s_{\text{crit}} \frac{\Delta v}{3 C_D \rho \text{gas} \Delta v} \sim \frac{16 c_s \rho_p s_{\text{crit}}}{3 C_D \Sigma \Delta v} \sim \frac{16 \rho_p s_{\text{crit}} a \Omega}{3 C_D \Sigma c_s} \]

Rearranging we get

\[
s_{\text{crit}} \sim \frac{3 C_D \Sigma c_s}{16 \rho_p a \Omega} \]

\[
3(1)(10^3 \text{ g cm}^{-2}) \sqrt{\left(\frac{k(300 \text{ K})}{2 m_p}\right)} \sim 16(1 \text{ g cm}^{-3})(1 \text{ AU}) \left(\frac{2\pi}{1 \text{ yr}}\right) \sim [7 \text{ cm}] \]

Since \( s_{\text{crit}} > \lambda_{mfp} \) this answer is ok.

(d) About how long does it take a critically sized particle to drift radially inwards due to drag? The particle starts at \( z = 0 \) at \( a = 1 \text{ AU} \), and we want to know how long it takes to reach, say, \( a = 0.5 \text{ AU} \). A numerical answer suffices.

The time for the particle to sink to the middle is the angular momentum, \( m v_{\text{orb}} a \), divided by
the torque, $F_D a$. Therefore we find

$$t_{\text{drift}} \sim \frac{m v_{\text{orb}}}{F_D} \sim \left(\frac{4\pi^3}{3} a^3 \rho a^2 \right) \left(\frac{\Omega a}{C_D \rho A \Delta v^2}\right)$$

$$\sim \frac{8 \rho_p s_{\text{crit}} \Omega a}{3 \rho_{\text{gas}} C_D \Delta v^2 a^3 \Omega^2} \sim \frac{16 \rho_p s_{\text{crit}} a^8 \Omega^2}{3 C_D \rho c_s^2 \Sigma a^2 \Omega^2} \sim \frac{a^2 \Omega}{c_s^2} \sim \frac{a}{\Delta v}$$

Plugging in for the values at 1 AU, we get

$$t_{\text{drift}} \sim \frac{1 \text{ AU}}{28 \text{ m s}^{-1}} \sim 170 \text{ years}$$

This is called the “drag crisis” problem of planet formation: how can one form planets when all the meter-sized rocks fall into the Sun in a century?