Problem 1. Post-Shock is Sub-Sonic

Shu Problem 4a, pages 449–450. We showed in class that post-shock material flows subsonically for strong shocks. This problem shows that the result holds true for any shock, weak or strong.

[15 points] We re-print here the problem:

Given the jump conditions for plane-parallel shocks,

\[ \rho_2 u_2 = \rho_1 u_1 \quad (1) \]
\[ P_2 + \rho_2 u_2^2 = P_1 + \rho_1 u_1^2 \quad (2) \]
\[ h_2 + \frac{u_2^2}{2} = h_1 + \frac{u_1^2}{2} \quad (3) \]

where \( h = \varepsilon + P/\rho = \gamma P/[(\gamma - 1)\rho] \) is the specific enthalpy of a perfect gas, derive the Prandtl-Meyer relation,

\[ u_1 u_2 = c_s^2 \quad (4) \]

where \( c_s^2 \) is given by

\[ \frac{\gamma + 1}{\gamma - 1} \frac{c_s^2}{2} = h + \frac{u^2}{2} \quad (5) \]

and equals a conserved quantity across the shock.

Hint: Manipulate the jump conditions into the form

\[ (u_1 - u_2) \left( \frac{c_s^2}{u_1 u_2} - 1 \right) = 0 \quad (6) \]

i.e., for \( u_1 \neq u_2, u_1 u_2 = c_s^2 \).

Prove, moreover, that

\[ c_1^2 \equiv \gamma P_1/\rho_1 < c_s^2 < c_2^2 \equiv \gamma P_2/\rho_2 \quad (7) \]

and that the Prandtl-Meyer relation requires, for a compressive shock, the upstream flow to be supersonic, \( u_1 > c_1 \), and the downstream flow to be subsonic, \( u_2 < c_2 \).

Problem 2. Blowing Bubbles

A star emits a wind of constant velocity \( v_w \) and mass-loss rate \( \dot{M} \). The wind impacts the surrounding interstellar medium of density \( \rho_0 \) and terminates in a shock (read: heliopause, or for the non-solar case, astrosphere).\(^1\)

\(^1\)See Wood et al. 2002, 2005 for impressive measurements of the astrospheres of other stars.
(a) [5 points] Argue your way to an expression for the radius $R$ of the shock as a function of time $t$, assuming that all of the mechanical energy of the wind is used to drive outwards the shocked (swept-up) ISM.

Hint: the mechanical (kinetic) energy deposited by the star into the ISM constantly grows with time. This is unlike a supernova which releases a fixed amount of energy within a short instant.

(b) [5 points] Zooming in on a small portion of the expanding spherical shock, we see that it behaves as a 1D plane-parallel shock, with fresh uncompressed gas entering the shock and compressed gas leaving the shock from behind.

What is the velocity of the flow just behind the shock, evaluated in the star’s rest frame (not following the shock)? Answer this question using the relevant jump condition(s) for a strong (Mach $\gg 1$) shock and $\gamma = 5/3$.

How can you reconcile this result with the fact that far behind the shock the flow velocity must be outward at speed $v_w$? Thereby deduce that there are actually two shocks—a “forward” shock whose radius $R$ you have solved for in (a), and a “reverse” shock whose radius $R_1 < R$. In the forward shock, the ISM gets shocked. In the reverse shock, the wind gets shocked.

Is the velocity you have just solved for appropriate for shocked ISM or shocked wind?

**OPTIONAL Problem 3. Bow Shock**

Consider two stars of masses $M_1$ and $M_2$ in a circular binary orbit. Star 1 emits a wind of $\dot{M}_1$ and constant speed $v_1$, and likewise for star 2. The two winds will collide in what is called a “bow shock”. The surface of the bow shock is defined by the condition that the momentum carried by star 1’s wind, measured perpendicular to the bow shock and across the bow shock, balances the momentum carried by star 2’s wind, also measured perpendicular to the bow shock and across the bow shock:

$$ (\rho_1 v_{1,\perp}) \times v_{1,\perp} = (\rho_2 v_{2,\perp}) \times v_{2,\perp} $$

Here we are ignoring the pressure of each wind (such winds are called “cold” flows), which is a fine approximation for O star winds.

As with the previous problem, what we are calling a single bow shock actually comprises 2 shocks—wind 1 gets shocked on the side of the bow shock closer to star 1, while wind 2 gets shocked on the side of the bow shock closer to star 2. For more details, see, e.g., Luo, McCray, and MacLow 1990, or Stevens, Blondin, & Pollack 1992.

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See the classic papers on this subject by Castor, McCray, and Weaver 1975 and Weaver et al. 1977.
(a) [3 points] Choose a frame rotating with the binary so that star 1 is fixed at the origin and star 2 is located a distance $D$ away on the x-axis. Within what distance of the origin can we ignore rotational forces? That is, within what distance of the binary can we approximate an individual wind as being a purely radial flow directed away from a star, with negligible Coriolis deflection?

Give an approximate analytic expression for this distance for the simple case $v_1 = v_2$ and $M_1 = M_2$. Also supply a numerical estimate for $v_1 = v_2 = 2000\text{ km s}^{-1}$, $M_1 = M_2 = 10M_\odot$, and $D = 10R_*$, where $R_* = 3R_\odot$ is the individual stellar radius. Express your numerical estimate in units of $D$.

(b) [6 points] Ignoring rotational effects, show that the local slope of the bow shock surface obeys

\[
\frac{dy}{dx} = \frac{R^{1/2}r_2 \sin \beta_1 + r_1 \sin \beta_2}{R^{1/2}r_2 \cos \beta_1 + r_1 \cos \beta_2} \tag{9}
\]

where $\vec{r}_1$ is the distance vector between star 1 and a given point on the bow shock, $\beta_1$ is the angle that $\vec{r}_1$ makes with respect to the x-axis, and similarly for 2-variables. Here the constant

\[
R = \frac{\dot{M}_1 v_1}{\dot{M}_2 v_2} \tag{10}
\]

(c) [3 points] Decide the location of the bow shock on the x-axis only, for $\dot{M}_1/\dot{M}_2 = 10$ and $v_1 = v_2$, in terms of $D$.

(d) [3 points] Draw the bow shock surface using (b) and (c).