Problem 1. Post-Shock is Sub-Sonic

Shu Problem 4a, pages 449–450

We start with the Rankine-Hugoniot jump conditions,

\[ \rho_1 u_1 = \rho_2 u_2 \]  
\[ \rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \]  
\[ \frac{1}{2} u_1^2 + h_1 = \frac{1}{2} u_2^2 + h_2 \]

and wish to prove that

\[ u_1 u_2 = c_s^2 \]  

by showing

\[ (u_1 - u_2) \left( \frac{c_s^2}{u_1 u_2} - 1 \right) = 0. \]

Here \( c_s^2 \) is given by

\[ \left( \frac{\gamma + 1}{\gamma - 1} \right) \frac{c_s^2}{2} = h + \frac{u^2}{2}, \]

where this quantity is conserved across the shock.

The specific enthalpy \( h = \gamma P / (\gamma - 1) \rho \), allows us to substitute \( P = (\gamma - 1) h \rho / \gamma \) into Eq. (2):

\[ \frac{\gamma - 1}{\gamma} h_2 \rho_2 + \rho_2 u_2^2 = \frac{\gamma - 1}{\gamma} h_1 \rho_1 + \rho_1 u_1^2. \]

Factoring out \( \frac{\gamma - 1}{\gamma} \rho \) from each side gives

\[ \left( \frac{\gamma - 1}{\gamma} \rho_2 \right) \left( h_2 + \frac{\gamma}{\gamma - 1} u_2^2 \right) = \left( \frac{\gamma - 1}{\gamma} \rho_1 \right) \left( h_1 + \frac{\gamma}{\gamma - 1} u_1^2 \right) \]

Rearranging Eq. (6) gives \( h = \left( \frac{\gamma + 1}{\gamma - 1} \right) \frac{c_s^2}{2} - \frac{u^2}{2} \). Plugging in for \( h_1 \) and \( h_2 \), and dividing each side by \( \frac{\gamma - 1}{\gamma} \rho_1 \), we obtain

\[ \frac{\rho_2}{\rho_1} \left( \frac{\gamma + 1}{\gamma - 1} \frac{c_s^2}{2} - \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} u_2^2 \right) = \frac{\gamma + 1}{\gamma - 1} \frac{c_s^2}{2} - \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} u_1^2, \]
which simplifies to
\[ \frac{\rho_2}{\rho_1} (c_s^2 + u_2^2) = c_s^2 + u_1^2. \]

Things are looking good now! Eq. (1) tells us that \( \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \), leading to
\[ \frac{u_1}{u_2} (c_s^2 + u_2^2) = c_s^2 + u_1^2. \]

Bringing all the terms to the left and dividing by \( u_1 \), and then some further massaging leads to:
\[ \frac{c_s^2}{u_2} - \frac{c_s^2}{u_1} - u_1 + u_2 = 0, \]
\[ \frac{c_s^2(u_1 - u_2)}{u_1 u_2} - (u_1 - u_2) = 0, \]
\[ (u_1 - u_2) \left( \frac{c_s^2}{u_1 u_2} - 1 \right) = 0, \]
which is precisely Eq. (5). Thus, for \( u_1 \neq u_2 \), we require that \( u_1 u_2 = c_s^2 \).

Next, we prove for a flow where \( u_1 > u_2 \) that
\[ a_1^2 \equiv \gamma P_1/\rho_1 < c_s^2 < a_2^2 \equiv \gamma P_2/\rho_2. \] (7)

Writing \( h_1 \) in terms of \( a_1 \) we have
\[ h_1 = \frac{\gamma P_1}{(\gamma - 1)\rho_1} = \frac{a_1^2}{\gamma - 1}. \] Plug this into Eq. (6) gives us
\[ \frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \left( \frac{\gamma + 1}{\gamma - 1} \right) \frac{c_s^2}{2} \]
\[ a_1^2 = \frac{c_s^2}{2} \left[ (\gamma + 2) + (\gamma - 1) \frac{u_1^2}{c_s^2} \right] \]
Using the Prandtl-Meyer relation, Eq. (4),
\[ \frac{a_1^2}{c_s^2} = \frac{1}{2} \left[ (\gamma + 1) - (\gamma - 1) \frac{u_1}{u_2} \right] \]
\[ \frac{a_1^2}{c_s^2} = \frac{1}{2} \left[ 2 + (\gamma - 1) - (\gamma - 1) \frac{u_1}{u_2} \right] \]
\[ \frac{a_1^2}{c_s^2} = 1 + \frac{\gamma - 1}{2} \left[ 1 - \frac{u_1}{u_2} \right] \]
Since we require \( \gamma \geq 1, \frac{u_1}{u_2} \geq 0 \). Then for \( u_1 > u_2 \) we have
\[ \frac{a_1^2}{c_s^2} < 1, \]
or,
\[ a_1^2 < c_s^2. \] (8)
Proceed similarly for $a_2$ to obtain
\[
\frac{a_2^2}{c_*^2} = 1 + \frac{\gamma - 1}{2} \left[ 1 - \frac{u_2}{u_1} \right]
\]
and
\[
\frac{a_2^2}{c_*^2} > 1. \tag{9}
\]
Relations (8) and (9) lead to (7),
\[
a_1^2 < c_*^2 < a_2^2.
\]
To show that the upstream flow is supersonic,
\[
a_1^2 < c_*^2 = u_1 u_2 < u_1^2
\]
or,
\[
a_1 < u_1.
\]
Similarly, the downstream flow is subsonic:
\[
a_2^2 > c_*^2 = u_1 u_2 > u_2^2
\]
or,
\[
a_2 > u_2.
\]

**Problem 2. Blowing Bubbles**

A star emits a wind of constant velocity $v_w$ and mass-loss rate $\dot{M}$. The wind impacts the surrounding interstellar medium of density $\rho_0$ and terminates in a shock (read: heliopause, or for the non-solar case, astrosphere).\(^1\)

(a) Argue your way to an expression for the radius $R$ of the shock as a function of time $t$, assuming that all of the mechanical energy of the wind is used to drive outwards the shocked (swept-up) ISM.

The star puts out mechanical power $(1/2)\dot{M}v_w^2$ over time $t$. Then the (kinetic) energy unleashed after time $t$ is $E = (1/2)\dot{M}v_w^2 t$. In the energy-conserving phase, all of this energy is used to drive outwards the shocked (swept-up) ISM. This shocked ISM has mass $M_{\text{ISM}} = (4/3)\pi\rho_0 R(t)^3$, and moves with velocity $\sim \dot{R}$. So we have
\[
E \sim \frac{1}{2} M_{\text{ISM}} \dot{R}^2 \tag{10}
\]
Dropping all the order unity constants (since this derivation is so crude that keeping the order unity constants would make us think we have an accuracy that we don’t really have), we find
\[
\rho_0 R^3 \dot{R}^2 \sim \dot{M}v_w^2 t \tag{11}
\]
Substituting in $\dot{R} \sim R/t$.

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\(^1\)See Wood et al. 2002, 2005 for impressive measurements of the astrospheres of other stars.
\[ R(t) \sim \left( \frac{\dot{M}v_{w}^{2}}{\rho_{0}} \right)^{1/5} t^{3/5}, \]

which is very similar to the Sedov-Taylor solution, except the index on time is 3/5 not 2/5, reflecting the continuous energy injection by the star (as opposed to the one-time-only deposition of energy for the bomb).

(b) What is the velocity of the flow just behind the shock, evaluated in the star’s rest frame (not following the shock)? Assume a strong shock and \( \gamma = 5/3 \).

How can you reconcile this result with the fact that far behind the shock the flow velocity must be outward at speed \( v_{w} \)? Thereby deduce that there are actually two shocks—a “forward” shock whose radius \( R \) you have solved for in (a), and a “reverse” shock whose radius \( R_{1} < R \).

In the forward shock, the ISM gets shocked. In the reverse shock, the wind gets shocked.

Is the velocity you have just solved for appropriate for shocked ISM or shocked wind?

In the frame of the shock (see Figure 1), gas comes into the shock with velocity \( \dot{R} \) and leaves with velocity \( \dot{R}/4 \) (for \( \gamma = 5/3 \) and a strong shock, \( u_{2}/u_{1} = (\gamma - 1)/(\gamma + 1) = 1/4 \), as derived in class. To transform to the lab frame, give every fluid parcel an outward radial velocity of \( \dot{R} \). So then the fluid in front of the shock is at rest (which is what we assumed), and the fluid just behind the shock moves radially outward with velocity \( 3\dot{R}/4 \) (Figure 2).

This velocity is very different from \( v_{w} \). There is no problem here, because \( 3\dot{R}/4 \) refers to the shocked ISM (which used to be at rest, and suddenly finds itself moving forward at \( 3\dot{R}/4 \) just after it gets shocked).

The stellar wind gets shocked, too, but not in the same shock that shocks the ISM. The stellar wind gets shocked in the “reverse” shock, whose radius \( R_{1} \) always trails \( R \). Solving for \( R_{1} \) is beyond the scope of this problem, but see the cited literature.

(c) The star, which has been blowing a wind for a long time, goes supernova. Argue your way to decide how the radius \( R_{s} \) of the supernova blast wave scales with time \( t \), assuming that all of the

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\(^2\)See the classic papers on this subject by Castor, McCray, and Weaver 1975 and Weaver et al. 1977.
mechanical energy of the supernova is used to drive outwards the shocked (swept-up) stellar wind. Just the scaling for this part is sufficient; no need to supply the coefficient.

Here, we should recognize that by conservation of mass, the constant wind creates a density gradient:

$$\dot{M} = 4\pi r^2 \rho v_w = \text{constant}$$

so that

$$\rho \propto \frac{1}{r^2}. \quad (14)$$

There’s the textbook answer and then the answer in real life. First the textbook answer. Repeat the Sedov-Taylor derivation, but with the supernova blast wave propagating into a medium with $\rho \propto 1/r^2$. Again, we have

$$E \sim \frac{1}{2} M \dot{R}^2 \quad (15)$$

So the swept-up stellar wind has a mass that grows as $M_{\text{shocked medium}} \propto \rho R^3 \propto (1/R^2) \times R^3 \propto R$. For $E = E_{\text{supernova}} = \text{constant}$ and $\dot{R} \sim R/t$, 

$$E_{\text{SN}} \sim M_{\text{shocked medium}} \dot{R}^2 \propto R \dot{R}^2 \propto R^3 t^{-2}, \quad (16)$$

or,

$$R \propto t^{2/3}. \quad (17)$$

Now the real life answer. The Sedov-Taylor-type solution only applies after the shocked, swept-up mass exceeds the original explosive mass. But for a typical supernova, that condition does not obtain—the wind mass might be a few solar masses or less, while the supernova ejecta mass might be several solar masses. If the supernova ejecta mass is much greater than the wind mass, then the ejecta mass doesn’t slow down until it encounters the ISM.

**Problem 3. Bow Shock**

Consider two stars of masses $M_1$ and $M_2$ in a circular binary orbit. Star 1 emits a wind of $\dot{M}_1$ and constant speed $v_1$, and likewise for star 2. The two winds will collide in what is called a “bow shock”. The surface of the bow shock is defined by the condition that the momentum carried by star 1’s wind, measured perpendicular to the bow shock and across the bow shock, balances the momentum carried by star 2’s wind, also measured perpendicular to the bow shock and across the bow shock:
Here we are ignoring the pressure of each wind (such winds are called “cold” flows), which is a fine approximation for O star winds.

As with the previous problem, what we are calling a single bow shock actually comprises 2 shocks—wind 1 gets shocked on the side of the bow shock closer to star 1, while wind 2 gets shocked on the side of the bow shock closer to star 2. For more details, see, e.g., Luo, McCray, and MacLow 1990, or Stevens, Blondin, & Pollack 1992.

(a) Choose a frame rotating with the binary so that star 1 is fixed at the origin and star 2 is located a distance $D$ away on the $x$-axis. Within what distance of the origin can we ignore rotational forces? That is, within what distance of the binary can we approximate an individual wind as being a purely radial flow directed away from a star?

Give an approximate analytic expression for this distance for the simple case $v_1 = v_2$ and $M_1 = M_2$. Also supply a numerical estimate for $v_1 = v_2 = 2000 \text{ km s}^{-1}$, $M_1 = M_2 = 10M_\odot$, and $D = 10R_*$, where $R_* = 3R_\odot$ is the individual stellar radius. Express your numerical estimate in units of $D$.

Let us first consider the effects of the Coriolis force. For a fluid parcel moving radially outward in the co-rotating frame, the Coriolis effect deflects our parcel perpendicular to this radial velocity, with an acceleration of $a_c = -2\Omega \times v$. The deflection due to Coriolis is

$$\Delta r \sim \frac{1}{2} a_c t^2 = \Omega v t^2 = \frac{\Omega r^2}{v}$$

We want this deflection to be small compared to $r$, so that $\Delta r/r < 1$, giving us $\frac{\Omega r}{v} < 1$ or

$$r < \frac{v}{\Omega}$$

Kepler’s law gives the angular frequency of the binary as

$$\Omega = \left( \frac{G(M_1 + M_2)}{D^3} \right)^{1/2}$$

Equations (20) and (21) combine to give

$$r < \frac{D^{3/2}}{\left[ G(M_1 + M_2) \right]^{1/2}}$$

We can also write this as $r < (v/v_{\text{orb}})D$, where $v_{\text{orb}}$ is the orbital velocity of the binary.

Proceed similarly for centrifugal effects. Using $a_{\text{cent}} = v^2/r = \Omega^2 r$, we get

$$\Delta r \sim \frac{1}{2} a_{\text{cent}} t^2 = \frac{1}{2} \Omega^2 r \left( \frac{r}{v} \right)^2 = \frac{\Omega^2 r^3}{2v^2}$$
Requiring that $\Delta r/r < 1$ gives $\Omega^2 r^2 < 1$, or

$$r < \frac{\sqrt{2}v}{\Omega},$$

which is greater than Equation (22) by a factor of $\sqrt{2}$, meaning Equation (22) supplies the limiting condition on $r$.

Plugging the prescribed conditions into (22) yields $r < 1.17 \times 10^{11}$ meters, or $r < 5.61D$. So, we are safe from rotational effects, since the bowshock is located between $M_1$ and $M_2$, at $r < D$.

(b) Ignoring rotational effects, show that the local slope of the bow shock surface obeys

$$\frac{dy}{dx} = \frac{R^{1/2}r_2 \sin \beta_1 + r_1 \sin \beta_2}{R^{1/2}r_2 \cos \beta_1 + r_1 \cos \beta_2}$$

where $\vec{r}_1$ is the distance vector between star 1 and a given point on the bow shock, $\beta_1$ is the angle that $\vec{r}_1$ makes with respect to the x-axis, and similarly for 2-variables. Here the constant

$$R = \frac{M_1v_1}{M_2v_2}$$

The statement of momentum balance, Equation (18), reads

$$\rho_1v_1^2 \sin^2(\theta - \beta_1) = \rho_2v_2^2 \sin^2(\beta_2 - \theta),$$

where angles are depicted in Figure 3. Continuity allows us to substitute in
\[ \rho_i v_i = \dot{M}_i / (4 \pi r_i^2), \] (28)

giving

\[
\frac{\dot{M}_1 v_1}{r_1^2} \sin^2(\theta - \beta_1) = \frac{\dot{M}_2 v_2}{r_2^2} \sin^2(\beta_2 - \theta)
\]

\[
\sqrt{\dot{M}_1 v_1 r_2 \sin(\theta - \beta_1)} = \sqrt{\dot{M}_2 v_2 r_1 \sin(\beta_2 - \theta)}
\]

Substituting \( R = \dot{M}_1 v_1 / \dot{M}_2 v_2 \) and employing some sum/difference trig formulas leads to

\[ r_2 R^{1/2} (\sin \theta \cos(-\beta_1) + \sin(-\beta_1) \cos \theta) = r_1 (\sin \beta_2 \cos(-\theta) + \sin(-\theta) \cos \beta_2) \]

Dividing through by \( \cos \theta \) and some rearranging leads to

\[ [r_2 R^{1/2} \cos \beta_1 + r_1 \cos \beta_2] \tan \theta = r_2 R^{1/2} \sin \beta_1 + r_1 \sin \beta_2 \]

The local slope \( \frac{dy}{dx} \) of the bow shock is given by \( \tan \theta \). Solving the above equation yields

\[
\frac{dy}{dx} = \tan \theta = \frac{r_2 R^{1/2} \sin \beta_1 + r_1 \sin \beta_2}{r_2 R^{1/2} \cos \beta_1 + r_1 \cos \beta_2},
\] (29)

which is what we originally wanted to show.

(c) Decide the location of the bow shock on the x-axis only, for \( \dot{M}_1 / \dot{M}_2 = 10 \) and \( v_1 = v_2 \), in terms of \( D \).

On the x-axis, winds are colliding head-on, so Equation (18) simplifies to \( \rho_1 v_1^2 = \rho_2 v_2^2 \). Substituting in continuity Equation (28),

\[
\frac{\dot{M}_1 v_1}{r_1^2} = \frac{\dot{M}_2 v_2}{r_2^2},
\] (30)
which leads to

\[ R \equiv \frac{\dot{M}_1 v_1}{M_2 v_2} = \frac{r_1}{r_2} = \frac{r_1^2}{(D - r_1)^2} \]  

(31)

(see Figure 4). This gives a quadratic equation in \( r_1 \):

\[ (R - 1)r_1^2 - (2RD)r_1 + RD^2 = 0 \]  

(32)

with roots

\[ r_1 = D \left[ \frac{R \pm \sqrt{R}}{R - 1} \right] \]  

(33)

For \( \dot{M}_1/\dot{M}_2 = 10 \) and \( v_1 = v_2 \), \( R = 10 \) (insert into Eqn. (26)). Choosing the negative root solution, since the shock has to lie at \( r_1 < D \), we get

\[ r_1 = D \left[ \frac{10 - \sqrt{10}}{10 - 1} \right] \approx 0.76D \]  

(34)

(d) Draw the bow shock surface using (b) and (c).

Guidelines for creating a bow shock figure:

Convert \( r_1, r_2, \beta_1, \beta_2 \) to \( x_1, x_2, y_1, y_2 \) via

\[ y_1 = r_1 \sin \beta_1 \]
\[ y_2 = r_2 \sin \beta_2 \]
\[ x_1 = r_1 \cos \beta_1 \]
\[ x_2 = r_2 \cos \beta_2 \]

Also, note that \( x_2 = x_1 - D \) and \( y_2 = y_1 \). Set \( D = 1 \) to get distances in units of \( D \). Using \( \frac{dy}{dx} \) from part (b) and the initial condition \( x = 0.76 \) for \( y = 0 \) from (c), you can plot the bow shock (see Figure 5).