Problem 1. Self-Similar Magic

In class we defined the dimensionless functions for pressure, density, and velocity behind a strong, spherical shock expanding into a uniform medium of density $\rho_1$:

$$
P(\xi) \equiv \frac{P(r,t)}{P_2(t)} = \frac{\gamma + 1}{2} \frac{P(r,t)}{\rho_1 |u_{sh}(t)|^2} \tag{1}
$$

$$
\alpha(\xi) \equiv \frac{\rho(r,t)}{\rho_2(t)} = \frac{\gamma - 1}{\gamma + 1} \frac{\rho(r,t)}{\rho_1} \tag{2}
$$

$$
v(\xi) \equiv \frac{u(r,t)}{u_2(t)} = \frac{\gamma + 1}{2} \frac{u(r,t)}{u_{sh}(t)} \tag{3}
$$

where the notation follows that of lecture (note the subtle font difference between the dimensionless $P$ and the dimensional $P$). The similarity variable $\xi \equiv r/R(t)$ where $R(t) = A(E/\rho_1)^{1/5}t^{2/5}$ is the radius of the shock front and $u_{sh} = dR/dt$.

(a) [15 points] Transform the PDEs for mass, momentum, and energy conservation (written down in class, and also expressed in equations 17.42abc of Thorne & Blandford) into a set of more manageable ODEs (equations 17.47abc of Thorne & Blandford):

$$
0 = 2\alpha v' - (\gamma + 1)\xi \alpha' + v(2\alpha' + \frac{4}{\xi} \alpha) \tag{4}
$$

$$
0 = \alpha v[3(\gamma + 1) - 4v'] + 2(\gamma + 1)\xi \alpha v' - 2(\gamma - 1)P' \tag{5}
$$

$$
3 = \left( \frac{2v}{\gamma + 1} - \xi \right) \left( \frac{P'}{P} - \gamma \frac{\alpha'}{\alpha} \right) \tag{6}
$$

where $' = d/d\xi$. The value gained here is that we no longer have two dimensions, $r$ and $t$, to worry about; we have only a single dimension, $\xi$, because $R(t)$ “magically” disappeared from the equations (more on magic in part b). The ODEs in this case are still daunting (and coupled and non-linear), but there are straightforward and robust procedures for solving ODEs (Runge-Kutta and Burlirsch-Stoer are a couple).

Equation (6) is probably best derived from $D(P/\rho^\gamma)/Dt = 0$, the form of the energy equation that we discussed in class, where $D/Dt$ is the convective derivative. (We discussed in class how this energy equation is not the same as $P/\rho^\gamma = \text{constant}$, and how the latter is not correct here because different fluid parcels cross the evolving shock front at different times and therefore pick up different entropies $K = P/\rho^\gamma$.)

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The necessary transformations of derivatives like $\partial/\partial r$ and $\partial/\partial t$ were derived in class and are also given in equations (17.45) and (17.46) of Thorne & Blandford, here reproduced for your convenience:

$$
\left( \frac{\partial}{\partial t} \right)_r = -\frac{2\xi}{5t} \left( \frac{\partial}{\partial \xi} \right)_R + \frac{2R}{5t} \left( \frac{\partial}{\partial R} \right)_\xi \tag{7}
$$

$$
\left( \frac{\partial}{\partial r} \right)_t = \frac{1}{R} \left( \frac{\partial}{\partial \xi} \right)_R \tag{8}
$$

[[NB: Shu defines the dimensionless functions $v(\xi)$ and $P(\xi)$ differently than we did in class. He inserts an extra $\xi$ into $v$ and an extra $\xi^2$ into $P$. It is not clear to me why he does this; the resulting equations look a bit nicer mathematically, I guess. By comparison, our definitions of $v$ and $P$ in class are the same as those of Thorne & Blandford (just with different notation; sorry; feel free to use whatever notation you want). ODEs result from either set of definitions, and the normalizations from either set of definitions are valid (e.g., $P(\xi = 1) = 1$ and $v(\xi = 1) = 1$ using either set of definitions.).]]

(b) [10 points] It may seem a bit magical that $R(t)$ vanishes from the equations, enabling us to transform a set of PDEs into a set of ODEs. Here we ask whether this transformation depended on our initial “inspired” order-of-magnitude scaling that $R(t) \propto t^{2/5}$, which was physically motivated by considering energy conservation.

Suppose more generally that $R \propto t^\beta$ instead (inspired by nothing more than the usual expectation among astronomers that “everything is a power law”). Is the transformation from PDEs into ODEs still possible without explicitly knowing $\beta$? What are some implications of your answer?