Astrophysical Fluid Dynamics – Solution Set 7
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Readings: Shu 230–240 (in class hand-out); however much of Thorne & Blandford Chapter 17 you like; selections from Sturrock in the Course Reader

Problem 1. Self-Similar Magic

In class we defined the dimensionless functions for pressure, density, and velocity behind a strong, spherical shock expanding into a uniform medium of density \( \rho_1 \):

\[
\mathcal{P}(\xi) \equiv \frac{P(r,t)}{P_2(t)} = \frac{\gamma + 1}{2} \frac{P(r,t)}{\rho_1[u_{sh}(t)]^2}
\]

\[
\alpha(\xi) \equiv \frac{\rho(r,t)}{\rho_2(t)} = \frac{\gamma - 1}{\gamma + 1} \frac{\rho(r,t)}{\rho_1}
\]

\[
v(\xi) \equiv \frac{u(r,t)}{u_2(t)} = \frac{\gamma + 1}{2} \frac{u(r,t)}{u_{sh}(t)}
\]

where the notation follows that of lecture (note the subtle font difference between the dimensionless \( \mathcal{P} \) and the dimensional \( P \)). The similarity variable \( \xi \equiv r/R(t) \) where \( R(t) = A(E/\rho_1)^{1/5} t^{2/5} \) is the radius of the shock front and \( u_{sh} = dR/dt \).

(a) Transform the PDEs for mass, momentum, and energy conservation (written down in class, and also expressed in equations 17.42abc of Thorne & Blandford) into a set of more manageable ODEs (equations 17.47abc of Thorne & Blandford):

\[
0 = 2\alpha v' - (\gamma + 1)\xi \alpha' + v(2\alpha' + 4\xi \alpha)
\]

\[
0 = \alpha v [3(\gamma + 1) - 4v'] + 2(\gamma + 1)\xi \alpha v' - 2(\gamma - 1)\mathcal{P}'
\]

\[
3 = \left( \frac{2v}{\gamma + 1 - \xi} \right) \left( \frac{\mathcal{P}'}{\mathcal{P}} - \frac{\gamma \alpha'}{\alpha} \right)
\]

where \( ' = d/d\xi \). The value gained here is that we no longer have two dimensions, \( r \) and \( t \), to worry about; we have only a single dimension, \( \xi \), because \( R(t) \) “magically” disappeared from the equations (more on magic in part b). The ODEs in this case are still daunting (and coupled and non-linear), but there are straightforward and robust procedures for solving ODEs (Runge-Kutta and Burlirsch-Stoer are a couple).

Equation (6) is probably best derived from \( D(P/\rho^2)/Dt = 0 \), the form of the energy equation that we discussed in class, where \( D/Dt \) is the convective derivative. (We discussed in class how this energy equation is not the same as \( P/\rho^2 \) = constant, and how the latter is not correct here because
different fluid parcels cross the evolving shock front at different times and therefore pick up different entropies $K = P/\rho^\gamma$.

The necessary transformations of derivatives like $\partial/\partial r$ and $\partial/\partial t$ were derived in class and are also given in equations (17.45) and (17.46) of Thorne & Blandford, here reproduced for your convenience:

\[
\left( \frac{\partial}{\partial t} \right)_r = -\frac{2\xi}{5t} \left( \frac{\partial}{\partial \xi} \right)_R + \frac{2R}{5t} \left( \frac{\partial}{\partial R} \right)_\xi
\]

\[
\left( \frac{\partial}{\partial r} \right)_t = \frac{1}{R} \left( \frac{\partial}{\partial \xi} \right)_R
\]

[[NB: Shu defines the dimensionless functions $v(\xi)$ and $P(\xi)$ differently than we did in class. He inserts an extra $\xi$ into $v$ and an extra $\xi^2$ into $P$. It is not clear to me why he does this; the resulting equations look a bit nicer mathematically, I guess. By comparison, our definitions of $v$ and $P$ in class are the same as those of Thorne & Blandford (just with different notation; sorry; feel free to use whatever notation you want). ODEs result from either set of definitions, and the normalizations from either set of definitions are valid (e.g., $P(\xi = 1) = 1$ and $v(\xi = 1) = 1$ using either set of definitions.).]]

(b) It may seem a bit magical that $R(t)$ vanishes from the equations, enabling us to transform a set of PDEs into a set of ODEs. Here we ask whether this transformation depended on our initial “inspired” order-of-magnitude scaling that $R(t) \propto t^{2/5}$, which was physically motivated by considering energy conservation.

Suppose more generally that $R \propto t^\beta$ instead (inspired by nothing more than the usual expectation among astronomers that “everything is a power law”). Is the transformation from PDEs into ODEs still possible without explicitly knowing $\beta$? What are some implications of your answer?

We will do (b) first and say $R(t) = ct^\beta$ (for some constant $c$), and then plug in $\beta = 2/5$ to do part (a). Much of this is plug and chug so we will just reproduce the energy equation (6); the other equations follow similarly.

Equation (7) for arbitrary $\beta$ becomes:

\[
\left( \frac{\partial}{\partial t} \right)_r = -\frac{\beta \xi}{t} \left( \frac{\partial}{\partial \xi} \right)_R + \frac{\beta R}{t} \left( \frac{\partial}{\partial R} \right)_\xi
\]

Then $D(P/\rho^\gamma)/Dt = (\partial/\partial t + u\partial/\partial r)(P/\rho^\gamma)$ becomes

\[
\left[ -\frac{\beta \xi}{t} \left. \frac{\partial}{\partial \xi} \right|_R + \frac{\beta R}{t} \left. \frac{\partial}{\partial R} \right|_\xi + \frac{2\beta R}{\gamma + 1} \frac{1}{t} \left. \frac{\partial}{\partial R} \right|_\xi \right] \left[ \frac{P}{\alpha^\gamma (\gamma + 1)^{\gamma - 1} \rho_1^\gamma} \right] = 0.
\]

Divide through by the constants to clear up the clutter:

\[
\left[ \left( -\xi + \frac{2v(\xi)}{\gamma + 1} \right) \left. \frac{\partial}{\partial \xi} \right|_R + R \left. \frac{\partial}{\partial R} \right|_\xi \right] \left[ \frac{P(\xi)}{\alpha(\xi)^\gamma} \right] = 0.
\]
Noting that \( t = (R/c)^{1/\beta} \):

\[
\left( \frac{2v}{\gamma + 1} - \xi \right) \left( \frac{R}{t} \right)^2 \left( \frac{\mathcal{P}'}{\mathcal{P}} - \gamma \frac{\alpha'}{\alpha} \right) + R \frac{\mathcal{P}}{\alpha^2} \frac{\partial}{\partial R} \left| _{\xi} \right. [R^2(R/c)^{-2/\beta}] = 0 \tag{12}
\]

Divide by \( \mathcal{P} \) and multiply by \( \alpha^2 \):

\[
\left( \frac{2v}{\gamma + 1} - \xi \right) \left( \frac{R}{t} \right)^2 \left( \frac{\mathcal{P}'}{\mathcal{P}} - \gamma \frac{\alpha'}{\alpha} \right) + R \frac{\partial}{\partial R} \left| _{\xi} \right. [R^2(R/c)^{-2/\beta}] = 0 \tag{13}
\]

\[
\left( \frac{2v}{\gamma + 1} - \xi \right) \left( \frac{R}{t} \right)^2 \left( \frac{\mathcal{P}'}{\mathcal{P}} - \gamma \frac{\alpha'}{\alpha} \right) + R \frac{e^{2/\beta} (2 - 2/\beta) R^{1-2/\beta}}{t^2} = 0 \tag{14}
\]

\[
\left( \frac{2v}{\gamma + 1} - \xi \right) \left( \frac{R}{t} \right)^2 \left( \frac{\mathcal{P}'}{\mathcal{P}} - \gamma \frac{\alpha'}{\alpha} \right) + (2 - 2/\beta) R^2/t^2 = 0 \tag{15}
\]

and so the \( R/t \) factor magically divides out:

\[
\left( \frac{2v}{\gamma + 1} - \xi \right) \left( \frac{R}{t} \right)^2 \left( \frac{\mathcal{P}'}{\mathcal{P}} - \gamma \frac{\alpha'}{\alpha} \right) + (2 - 2/\beta) = 0 \tag{16}
\]

which for \( \beta = 2/5 \) (the case considered in part (a)) reproduces equation (6) as desired. The other equations follow similarly.

So the answer to part (b) is \( \text{yes} \); it is possible to transform PDEs into ODEs without having to know \( \beta \). But the precise value of \( \beta \) determines the physical conditions of the explosion. The Sedov-Taylor \( \beta = 2/5 \) corresponds to the case of a spherical, energy-conserving shock propagating into a uniform medium. Other \( \beta \)'s correspond to shocks propagating under different physical conditions. In the “Blowing Bubbles” problem on a previous problem set, we found that when the shock is continuously injected with energy, then \( \beta = 3/5 \); we also found that when the shock propagated down a \( 1/r^2 \) density gradient, then \( \beta = 2/3 \). So it appears that we can use the same similarity formalism underlying Sedov-Taylor to write down ODEs corresponding to these other cases, just by changing \( \beta \).

**Problem 2. Drift**

Consider uniform electric \( \vec{E} \) and magnetic \( \vec{B} \) fields each oriented perpendicular to the other.

Show that the motion of a charged particle can be decomposed into a fast gyromotion about \( \vec{B} \) plus a slow drift at velocity

\[
\vec{v}_D = c \frac{\vec{E} \times \vec{B}}{B^2} \tag{17}
\]

This is a standard result that can be found in many textbooks. Please provide a complete derivation and also sketch the motions of both an electron and a proton, annotating your sketch with directions and magnitudes of motion.
Let’s consider the forces on the particle.

\[ \vec{F} = m\vec{a} = q\vec{E} + \frac{q}{c} \vec{v} \times \vec{B} \]

We can define our coordinates such that

\[ \vec{E} = E\hat{x} \]

and

\[ \vec{B} = B\hat{z} \]

So

\[ a_x = \frac{q}{m} E + \frac{q}{mc} Bv_y \]
\[ a_y = -\frac{q}{mc} Bv_x \]
\[ a_z = 0 \]

Now \( a = \dot{v} \), so we see from the last \( z \)-equation that a particle can undergo free motion parallel to the magnetic field.

To solve the other two equations, we’ll need to take some more time derivatives:

\[ \dot{a}_x = \ddot{v}_x = \frac{qB}{mc} \dot{v}_y = -\left( \frac{qB}{mc} \right)^2 v_x \]

and the solution to this differential equation is

\[ v_x = v_0 \sin \left( \frac{qB}{mc} t \right) \]

where I’ll be neglecting the arbitrary phase that would serve to meet some initial condition. Plugging this into the equation for \( a_y \) above gives

\[ a_y = -v_0 \frac{qB}{mc} \sin \left( \frac{qB}{mc} t \right) \]

which upon integrating gives

\[ v_y = v_0 \cos \left( \frac{qB}{mc} t \right) \]

Our solutions for \( v_x \) and \( v_y \) say that a charged particle executes circular motion (gyromotion) in the \( x-y \) plane (perpendicular to \( \vec{B} \)) with angular frequency \( qB/mc \). The handedness of the gyromotion depends on the sign of the charge of the particle, and the radius of the gyromotion, given here by \( mc v_0 / qB \), depends on the velocity and mass of the particle. This is the standard cyclotron motion.

Might there be any unaccelerated motion aside from that parallel to the magnetic field? Let’s try setting the total force \( \vec{F} \) equal to zero:

\[ \vec{E} = \frac{1}{c} \vec{B} \times \vec{v} \]

In other words, a charged particle moving with this velocity \( \vec{v} \) would experience zero net force — it would coast (drift) at this constant \( \vec{v} \). Let’s cross this into \( \vec{B} \). That yields

\[ \vec{E} \times \vec{B} = \frac{1}{c} (\vec{B} \times \vec{v}) \times \vec{B} = \frac{1}{c} (\vec{v}B^2 - \vec{B}(\vec{v} \cdot \vec{B})) \]
We’ve already considered motion parallel to the magnetic field and it’s not terribly interesting, so let’s consider motion perpendicular to the magnetic field, such that \( \vec{v} \cdot \vec{B} = 0 \). Then we have

\[
 c(\vec{E} \times \vec{B}) = \vec{v}B^2
\]

or

\[
 \vec{v}_d = c\frac{\vec{E} \times \vec{B}}{B^2}
\]

Therefore superposed on the gyromotion, there is an extra drift perpendicular to both the electric and magnetic fields. Notice that the drift points in the same direction \emph{regardless} of the sign of the charge, and regardless of the velocity and mass of the particle. Therefore, the total motion looks like a chain of loops (see Figures 1 and 2 for sketches of the motion). The amplitude of the gyromotion depends on the arbitrary constant \( v_0 \) (we can regard this as as a kind of “free oscillation”), but the drift velocity is independent of the properties of the particle and depends only on the magnitudes of the crossed fields.

It may seem a little magical that this drift velocity is independent of the particle properties. But this is just a consequence of the frame-dependent properties of electric and magnetic fields. If we boost into the frame moving at the drift velocity, then the particle doesn’t drift, it just executes cyclotron motion. In this rest frame of the fluid, we pick up an extra electric field equal to \( +\vec{v}_d \times \vec{B}/c = -\vec{E} \) from the non-relativistic Lorentz transform of \( \vec{B} \). This new electric field exactly cancels the original lab-frame electric field of \( \vec{E} \) so that

\[
\text{the total electric field seen in the rest frame drifting with the fluid is zero.}
\]

This is an equivalent statement to the ideal MHD flux-freezing statement.
Figure 1: $E \times B$ drift for a proton and an electron. The electron appears as a blue line and its gyromotion is too small to see here; see instead Figure 2. The drift velocity for both particles is in the $-y$ direction.
Figure 2: Same as Figure 1, but zoomed in to show the electron.