Astrophysical Fluid Dynamics – Problem Set 8

Readings: All chapters from Sturrock's Plasma Physics reprinted in the Course Reader; reprint from Wardle on non-ideal MHD in the Course Reader; Shu pages 338 to the end of the first complete paragraph on page 341, and pages 360–365 on ambipolar diffusion

Problem 1. Generalized Ohm's Law (Three Kinds of Non-Ideal MHD)

Consider a fluid that is composed predominantly of neutrals, with a small mixture of ions and electrons (cold star-forming molecular clouds and planet-forming disks are examples of such lightly ionized gases). Each ion and electron has the same absolute magnitude of charge $e\,(>0)$. The mass of an ion is m_i, the mass of an electron is m_e, and the number density of ions equals the number density of electrons, $n_i = n_e$ (net charge neutrality). The fluid is threaded with a magnetic field $\vec{B}$. Much of this problem was inspired by papers by Wardle (e.g., Wardle 1999; Wardle & Ng 1999; Salmeron & Wardle 2003; Wardle 2007) which you are free to look up; Wardle 2007 is in the Course Reader.

The neutrals (each of mass m with no subscript; e.g., H atoms, or H_2 molecules) provide the bulk of the resistance (a.k.a. collisional drag) against the flow of ions and electrons. That is, collisions between ions and electrons (considered in class) are negligible here compared to collisions between ions and neutrals, and between electrons and neutrals.

In class, when discussing the collisional drag force between two fluids, we considered the combination $\mu\vec{u}_{\rm rel}\nu$ [units of momentum per time = force], where ν was the collision rate and $\mu\vec{u}_{\rm rel}$ was the momentum exchanged per collision, with μ equal to the reduced mass between colliding partners, and $\vec{u}_{\rm rel}$ equal to the mean relative velocity. For ions colliding with neutrals, we have $\mu_i\vec{u}_{\rm i}\nu_i$, where $\mu_i = mm_i/(m+m_i)$, $\vec{u}_{\rm i}$ is the mean RELATIVE velocity between ions and neutrals,[1] and ν_i is the collision rate of a single ion in a sea of neutrals.

The literature on lightly ionized fluids (including Shu) uses different notation, which we will also use here. For ions colliding with neutrals, the literature replaces our $\mu_i\nu_i\vec{u}_{\rm i}$ with the equivalent $\rho m_i\gamma_i\vec{u}_{\rm i}$, where ρ is the mass density of neutrals and γ_i is a collisional rate coefficient. An expression for γ_i is given in Shu (page 362; γ_i is equally good for ions colliding with neutrals and neutrals colliding with ions). The new notation is purely a matter of convention. The advantage of this notation is that it gives a simple and sensible expression for the mean free time for a given ion to collide in a sea of neutrals (i.e., the time for the ion fluid to lose its mean momentum relative to neutrals): $t_i = [\text{momentum}]/[\text{force}] = m_i\vec{u}_{\rm i}/(\rho m_i\gamma_i\vec{u}_{\rm i}) = 1/(\rho\gamma_i)$. The corresponding single-ion collision frequency is just $1/t_i = \rho\gamma_i$. Analogous statements apply for an electron colliding with a sea of neutrals (just replace the subscript i with e).

[1] Our notation here for $\vec{u}_i$ differs a bit from that in class. In class, $\vec{u}_i$ was the velocity of ions measured in the lab frame. Here $\vec{u}_i$ (unprimed) is the velocity of ions measured RELATIVE TO THE NEUTRAL FLUID. I did not want to introduce primes for $\vec{u}_i$ even though it is the velocity of the ions in the (predominantly neutral) fluid rest frame; if we did, we would end up writing a ton of primes everywhere (as you'll see when you do the problem). Not priming $\vec{u}_i$ also seems acceptable because the relative velocity between two fluids (here ions and neutrals) does not change between the lab frame and the fluid rest frame.

The *Hall parameter* is the ratio of the particle gyrofrequency (a.k.a. cyclotron frequency) to the single-particle collision frequency:

$$\beta_i \equiv \frac{eB}{m_i c}\frac{1}{\rho\gamma_i} \tag{1}$$

$$\beta_e \equiv \frac{eB}{m_e c}\frac{1}{\rho\gamma_e} \tag{2}$$

where c is the speed of light and B is the magnitude of the magnetic field. A particle with $\beta \gg 1$ is said to be "tied to the field": it can gyrate many times around a field line before it gets perturbed by a neutral. Generally $\beta_e \gg \beta_i$—although not by the full factor of $m_i/m_e \sim 42^2$, since there are mass dependences in γ_i and γ_e (i.e., electrons have random thermal velocities that are larger than ion thermal velocities which makes $\gamma_e/\gamma_i > 1$).

This problem explores how the electrical conductivity of the fluid changes in the limits of high and low β, and by extension how the induction equation for $\partial\vec{B}/\partial t$ changes.

Many parts of this problem can be done independently of one another. For example, one does not need to solve (a) (which is a bit involved) to solve other parts. Also note that each part may ask several questions.

(a) [10 points] In class we stated that in many situations the equation of motion of a charged particle is dominated by electromagnetic forces and collisional drag. These forces typically overwhelm gravity, pressure gradients, and inertial forces (whether this is true or not in any given situation can be tested to order-of-magnitude).

In the frame co-moving with the neutrals,

$$+e(\vec{E}' + \vec{u}_i \times \vec{B}/c) - \rho\gamma_i m_i \vec{u}_i = 0 \tag{3}$$

$$-e(\vec{E}' + \vec{u}_e \times \vec{B}/c) - \rho\gamma_e m_e \vec{u}_e = 0 \tag{4}$$

where $\vec{u}_i$ and $\vec{u}_e$ are the (bulk) velocities of ions and electrons RELATIVE TO THE NEUTRALS (see previous footnote). The electric field $\vec{E}'$ is primed to remind us that this is the field seen in the rest frame of the (overwhelmingly) neutral fluid; later we will switch back to the unprimed lab frame. We WON'T prime the magnetic field $\vec{B}$ to remind us that the magnetic field does NOT change between frames—assuming non-relativistic speeds.

Use the ion and electron equations of motion to derive a "generalized Ohm's Law" connecting the current density $\vec{j} = n_i e\vec{u}_i - n_e e\vec{u}_e$, the components of the electric field that are parallel ($\vec{E}'_{\parallel}$) and perpendicular ($\vec{E}'_{\perp}$) to the magnetic field, and $\hat{B}$ (the unit vector pointing in the direction of the magnetic field):

$$\vec{j} = \sigma_{\parallel}\vec{E}'_{\parallel} + \sigma_H \hat{B} \times \vec{E}'_{\perp} + \sigma_P \vec{E}'_{\perp} \tag{5}$$

where the electrical conductivity parallel to the $\vec{B}$ field is

$$\sigma_{\parallel} = \frac{ec}{B}(n_i\beta_i + n_e\beta_e)\,; \tag{6}$$

the Hall conductivity

$$\sigma_H = -\frac{ec}{B}\left(\frac{n_i\beta_i^2}{1+\beta_i^2} - \frac{n_e\beta_e^2}{1+\beta_e^2}\right) = \frac{ec}{B}\left(\frac{n_i}{1+\beta_i^2} - \frac{n_e}{1+\beta_e^2}\right); \tag{7}$$

and the Pedersen conductivity

$$\sigma_P = \frac{ec}{B}\left(\frac{n_i\beta_i}{1+\beta_i^2} + \frac{n_e\beta_e}{1+\beta_e^2}\right). \tag{8}$$

The second equality of equation (7) is derived from the first equality using charge neutrality $n_i = n_e$.

Hint: I was able to simplify the algebra by taking, without loss of generality, $\vec{B} = [0, 0, B]$ and $\vec{E}' = [E'_\perp, 0, E'_\parallel]$. Then use equation (3) to solve for $\vec{u}_i = [u_{i1}, u_{i2}, u_{i3}]$, likewise equation (4) for the electrons, and use these to find $\vec{j}$. Finally cast your answer in the coordinate-independent language of equation (5) by recalling your coordinate choices for $\parallel$ and $\perp$.

(b) [10 points] **The low-β limit**: Suppose $\beta_i \ll \beta_e \ll 1$ (charged particles are not necessarily well-tied to the field because collisions dominate).[2]

First decide which cross-field current is more important. Which is larger, the second or the third term on the RHS of equation (5)? Drop the smaller term.

Examine the relative magnitudes of the conductivities of the two terms that you have left, working in the low $\beta_i \ll \beta_e \ll 1$ limit. Simplify accordingly as much as you can, and solve for $\vec{E}'$ in terms of $\vec{j}$.

In the lab frame, the velocity of the neutrals is $\vec{u}$.[3] Write down $\vec{E}$, the electric field seen in the lab frame, in terms of $\vec{E}'$, $\vec{u}$, and other given quantities.

Plug your expression for $\vec{E}$ into Maxwell's induction equation:

$$\frac{\partial \vec{B}}{\partial t} = -c\nabla \times \vec{E}\,. \tag{9}$$

Replace $\vec{j}$ with Ampere's law:

$$\nabla \times \vec{B} = 4\pi\vec{j}/c \tag{10}$$

[2] Even if $\beta_i, \beta_e \ll 1$, it is still possible that flux freezing can hold. It depends on the length scale. On large enough length scales — i.e., for large enough magnetic Reynolds number (see part c) — flux freezing can hold. Collisions only knock charged particles off field lines on small scales. Moreover, collisions are actually necessary for coupling the charged particles to the neutrals — otherwise the neutrals don't participate in the magnetized flow, which is what happens with ambipolar diffusion (parts d, e, and f).

[3] This velocity $\vec{u}$ (no subscript) is a lab-frame velocity, of the neutrals only. It should not be confused with $\vec{u}_i$ and $\vec{u}_e$, which for this problem are ion and electron velocities RELATIVE to the neutrals.

and use the vector identity

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} \tag{11}$$

and the fact that there are no magnetic monopoles (in this universe)

$$\nabla \cdot \vec{B} = 0 \tag{12}$$

to write down a formula for $\partial \vec{B}/\partial t$ in terms of $\vec{u}$ and $\vec{B}$ and whatever conductivities and constants you need.

This low β limit represents the regime of *Ohmic diffusion*. Is Ohmic diffusion important in high density or low density environments?

(c) [2 points] Define a "magnetic Reynolds number" by comparing the order of magnitude of the flux-freezing term (put this in the numerator) to the order of magnitude of the Ohmic diffusion term (put this in the denominator). Show that a sensible definition is:

$$Re_M \equiv \frac{Lu}{\eta} \tag{13}$$

where $\eta \equiv c^2/(4\pi\sigma)$ is the magnetic diffusivity, and L and u are characteristic length and velocity scales for the problem. High Re_M flows behave in the ideal MHD limit where magnetic flux is conserved; for low Re_M flows, the magnetic field dissipates by diffusing away.

(d) [10 points] **The high β limit**: Now assume $\beta_e \gg \beta_i \gg 1$ (particles are well-tied to the field).

As in part (b), first decide which cross-field current is more important. Which is larger, the second or the third term on the RHS of equation (5)? Drop the smaller term.

Now assume the remaining two contributions to the current density are comparable in magnitude. Based on the relative magnitudes of the relevant conductivities, decide which electric field component is larger, $E'_{\parallel}$ or $E'_{\perp}$. Thus show that (it is sufficient to show that the equation is true):

$$\vec{E}' \simeq -\frac{(\vec{j} \times \vec{B}) \times \vec{B}}{c^2 \rho \rho_i \gamma_i} \,. \tag{14}$$

where ρ_i is the mass density of ions. (Hint: only one component of $\vec{j}$ will survive the cross product above.)

As in (b), Lorentz transform back to the lab frame to find $\vec{E}$, replace $\vec{j}$ with $\vec{B}$ using Ampere's law, and insert into the induction equation to write down a formula for $\partial \vec{B}/\partial t$ in terms of $\vec{u}$ and $\vec{B}$ (you may compare your answer to equation 27.12 of Shu).

This is the induction equation with a new non-ideal term: that for *ambipolar diffusion*. Insofar as it involves two spatial derivatives, this non-ideal term also acts to diffuse away magnetic field, like Ohmic diffusion. But unlike Ohmic diffusion, the diffusivity depends on the quantity being diffused; the diffusivity scales as B^2 (see also Shu page 364). So roughly speaking, where B is stronger, the field diffuses away faster.

[Optional (no extra points): Make the connection with diffusion a bit clearer by dispensing with the double cross products in equation (14) and keeping $\vec{E}' \simeq \vec{j}_\perp/\sigma_P = (c/4\pi)(\nabla \times \vec{B})_\perp/\sigma_P$ where we have used Ampere's law. Thus write down:

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left[\frac{c^2}{4\pi\sigma_P} (\nabla \times \vec{B})_\perp \right] + \nabla \times (\vec{u} \times \vec{B}) \tag{15}$$

The diffusion-like term (first term on the rhs) is similar to the Ohmic diffusion term (insofar as both have two spatial derivatives), with σ replaced by $\sigma_P \propto 1/B^2$; the magnetic diffusivity $c^2/(4\pi\sigma_P) \propto B^2$ increases with the magnetic field strength. Notice we have not taken σ_P out of the curl because it depends on B. Compare with the ambipolar diffusion term in equation (28) of Wardle (2007).]

(e) [3 points] In ambipolar diffusion, the Hall parameters β_e and β_i are both $\gg 1$, which means both electrons and ions are well-tied to the field. So when the field diffuses away, it carries both electrons and ions with it—hence the term "ambipolar," meaning of both polarities. The magnetized charged plasma diffuses away and leaves the neutrals behind. Ambipolar diffusion is necessary for, e.g., star formation (enables neutral gas to slip out of field).

We can estimate the relative ion-neutral slip speed—recall we have been calling this u_i—as follows. Return to the ion and electron equations of motion (3) and (4) to derive:

$$\frac{\vec{j} \times \vec{B}}{c} = \rho_i \rho \gamma_i \vec{u}_i \tag{16}$$

where we have dropped the electron drag term which is small compared to the ion drag term (mostly because $\mu_i \gg \mu_e$, the fact that $\gamma_e > \gamma_i$ notwithstanding; also $u_i \sim u_e$ because both are well-tied to the field, notwithstanding the usual tiny ion-electron slip velocity that generates current).

Use Ampere's law again to find that, to order-of-magnitude,

$$u_i \sim \frac{B^2}{4\pi \rho_i \rho \gamma_i L} \tag{17}$$

There are astrophysical systems where this drift velocity can be large, e.g., comparable to sound speeds (e.g., Wang & Goodman 2017).

(f) [3 points] By analogy to how we defined the dimensionless magnetic Reynolds number Re_M in (c), define another dimensionless number, Am, that compares the magnitude of the flux-freezing term to the ambipolar diffusion term in the induction equation that you solved for in part (d). Show that to order-of-magnitude,

$$Am = \frac{u^2}{v_A^2}\frac{t}{t_{ni}} \tag{18}$$

where the Alfven speed $v_A = \sqrt{B^2/4\pi\rho}$ (where ρ is the density of neutrals), $t = L/u$ is a characteristic timescale in the flow, and $t_{ni} = 1/(\rho_i\gamma_i)$ is the timescale for a single neutral to collide with an ion in a sea of ions. Thus conclude that if $u \sim v_A$ (true if ram pressures ρu^2 are comparable to magnetic pressures B^2), then $Am \ll 1$ (flux freezing fails and ambipolar diffusion is important) if $t_{ni} \gg t$. This makes sense—if a neutral can't find an ion to collide and share momentum with before the flow changes, then the neutral gets left behind (and notice how we are now talking about a given neutral in a sea of ions, not a given ion in a sea of neutrals—the latter is controlled by the ion Hall parameter).

Bottom line: is ambipolar diffusion significant in low density or high density environments?

(g) [2 points] There is a third non-ideal regime. What is true about β_e and β_i here? This is called the *Hall diffusion* regime.