Problem 1. $\vec{E} \times \vec{B}$ Drift

Consider uniform electric $\vec{E}$ and magnetic $\vec{B}$ fields each oriented perpendicular to the other.

[10 points] Show that the motion of a charged particle can be decomposed into a fast gyromotion about $\vec{B}$ plus a slow drift at velocity

$$\vec{v}_D = c \frac{\vec{E} \times \vec{B}}{B^2}$$

Remarkably, $\vec{v}_D$ is independent of the sign and magnitude of the particle’s charge. This is a standard result that can be found in many textbooks (which you are free to look at). Please provide a complete derivation and also sketch the motions of both an electron and a proton, annotating your sketch with directions and magnitudes of motion.

Let’s consider the forces on the particle.

$$\vec{F} = m\vec{a} = q\vec{E} + \frac{q}{c} \vec{v} \times \vec{B}$$

We can define our coordinates such that

$$\vec{E} = E\hat{x}$$

and

$$\vec{B} = B\hat{z}$$

So

$$a_x = \frac{q}{m} E + \frac{q}{mc} Bv_y$$

$$a_y = -\frac{q}{mc} Bv_x$$

$$a_z = 0$$

Now $a = \dot{v}$, so we see from the last $z$-equation that a particle can undergo free motion parallel to the magnetic field.

To solve the other two equations, we’ll need to take some more time derivatives:

$$\dot{a}_x = \dot{v}_x = \frac{qB}{mc} v_y = -\left(\frac{qB}{mc}\right)^2 v_x$$

and the solution to this differential equation is

$$v_x = v_0 \sin \left(\frac{qB}{mc} t\right)$$
where I’ll be neglecting the arbitrary phase that would serve to meet some initial condition. Plugging this into the equation for $a_y$ above gives

$$a_y = -v_0 \frac{qB}{mc} \sin \left(\frac{qB}{mc}t\right)$$

which upon integrating gives

$$v_y = v_0 \cos \left(\frac{qB}{mc}t\right)$$

Our solutions for $v_x$ and $v_y$ say that a charged particle executes circular motion (gyromotion) in the $x$-$y$ plane (perpendicular to $\vec{B}$) with angular frequency $qB/mc$. The handedness of the gyromotion depends on the sign of the charge of the particle, and the radius of the gyromotion, given here by $\frac{mcv_0}{qB}$, depends on the velocity and mass of the particle. This is the standard cyclotron motion.

Might there be any unaccelerated motion aside from that parallel to the magnetic field? Let’s try setting the total force $\vec{F}$ equal to zero:

$$\vec{E} = \frac{1}{c} \vec{B} \times \vec{v}$$

In other words, a charged particle moving with this velocity $\vec{v}$ would experience zero net force — it would coast (drift) at this constant $\vec{v}$. Let’s cross this into $\vec{B}$. That yields

$$\vec{E} \times \vec{B} = \frac{1}{c} (\vec{B} \times \vec{v}) \times \vec{B} = \frac{1}{c} (\vec{v}B^2 - \vec{B}(\vec{v} \cdot \vec{B})))$$

We’ve already considered motion parallel to the magnetic field and it’s not terribly interesting, so let’s consider motion perpendicular to the magnetic field, such that $\vec{v} \cdot \vec{B} = 0$. Then we have

$$c(\vec{E} \times \vec{B}) = \vec{v}B^2$$

or

$$\vec{v}_d = c \frac{\vec{E} \times \vec{B}}{B^2}$$

Therefore superposed on the gyromotion, there is an extra drift perpendicular to both the electric and magnetic fields. Notice that the drift points in the same direction regardless of the sign of the charge, and regardless of the velocity and mass of the particle. Therefore, the total motion looks like a chain of loops (see Figures 1 and 2 for sketches of the motion). The amplitude of the gyromotion depends on the arbitrary constant $v_0$ (we can regard this as as a kind of “free oscillation”), but the drift velocity is independent of the properties of the particle and depends only on the magnitudes of the crossed fields.

It may seem a little magical that this drift velocity is independent of the particle properties. But this is just a consequence of the frame-dependent properties of electric and magnetic fields. If we boost into the frame moving at the drift velocity, then the particle doesn’t drift, it just executes cyclotron motion. In this rest frame of the fluid, we pick up an extra electric field equal to $+\vec{v}_d \times \vec{B}/c = -\vec{E}$ from the non-relativistic Lorentz transform of $\vec{B}$. This new electric field exactly cancels the original lab-frame electric field of $\vec{E}$ so that the total electric field seen in the rest frame drifting with the fluid is zero. This is an equivalent statement to the ideal MHD flux-freezing statement.
Figure 1: $E \times B$ drift for a proton and an electron. The electron appears as a blue line and its gyromotion is too small to see here; see instead Figure 2. The drift velocity for both particles is in the -$y$ direction.
Figure 2: Same as Figure 1, but zoomed in to show the electron.
Problem 2. Rigidly Rotating Magnetospheres

The magnetosphere is the immediate environment of a spinning, magnetized star, where the magnetic field dominates all other sources of energy density. This problem considers how plasma within the magnetosphere moves, under steady, ideal-MHD conditions.

Work in cylindrical coordinates \((d, \phi, z)\), where \(d\) measures distance from the rotation axis of the star, and in the lab frame where the star is observed to be spinning. Note that at this stage of the problem, the star and its atmosphere/magnetosphere may, in principle, be spinning in a complicated way—the angular frequency \(\vec{\Omega} = \Omega \hat{z}\) may vary with position (see part e). This problem will show that \(\vec{\Omega}\) is actually constrained by the magnetic field.

(a) [5 points] Assuming infinite electrical conductivity, write down an expression for the electric field \(\vec{E}\) seen in the lab frame, in terms of the velocity of the plasma \(\vec{v}\) and the magnetic field \(\vec{B}\). At this point \(\vec{v}\) and \(\vec{B}\) are completely arbitrary.

We know that
\[
\vec{j} = \sigma (\vec{E} + \vec{v} \times \vec{B}/c)
\]
We’re given that the conductivity is effectively infinite. The current had better be finite. Thus, the term in parentheses must be zero and \(\vec{E} = -\vec{v} \times \vec{B}/c\).

(b) [5 points] Now take the magnetosphere to be in steady state. This is actually a very restrictive condition, as the rest of the problem will show. Use Faraday’s law of induction to prove that \(E_\phi = 0\).

One way to do this is to use the integral form of Faraday’s law,
\[
\frac{\partial \Phi}{\partial t} = \oint \vec{E} \cdot d\vec{l}
\]
where the integral is a line integral along a loop, and \(\Phi\) is the magnetic flux enclosed by the loop. Since \(\partial \Phi/\partial t = 0\) by the steady-state assumption, then the line integral is zero for every loop you can possibly draw. In particular, you can draw circular loops that encircle the star, in which case the integrand involves \(E_\phi\). Since the line integral is zero for every conceivable loop, the only way this can be true is if \(E_\phi = 0\).

(c) [5 points] Combine (a) and (b) to deduce that the plasma’s poloidal (non-\(\phi\)) velocity is parallel to the poloidal magnetic field. That is, show that \(B_z/B_d = v_z/v_d\).

The poloidal field is the \((d, z)\) component of the field; e.g., in a dipole field, a single poloidal field line traces out a big “ear” on the star.\(^1\) This part (c) shows that gas parcels slide up and down a poloidal field line like beads sliding along a curved wire.

\(^1\)Draw two such poloidal field lines on either limb of the star and you are on your way to drawing Dumbo the elephant.
Combining $E_\phi = 0$ and $\vec{E} = -\vec{v} \times \vec{B}/c$ gives

$$(\vec{v} \times \vec{B})_\phi = B_z v_d - B_d v_z = 0$$

which implies that $B_z/B_d = v_z/v_d$, which means that the plasma’s poloidal velocity (in the d-z plane at fixed $\phi$) is perfectly parallel to the poloidal magnetic field.

(d) [5 points] Use the ideal MHD induction equation for $\partial \vec{B}/\partial t$, the absence of magnetic monopoles, and the assumption that the flow is incompressible to find that

$$\frac{\partial \vec{B}}{\partial t} = (\vec{B} \cdot \nabla)\vec{v} - (\vec{v} \cdot \nabla)\vec{B}$$

(Aside: we saw earlier in the course that subsonic flows were nearly incompressible; in subsonic flows, the ram pressure is less than the thermal pressure. The generalization of that result is that if flow speeds are subsonic and sub-Alfvénic, then flow incompressibility holds; in sub-Alfvénic flows, the ram pressure is less than the magnetic pressure. We will study Alfvén waves later.)

The induction equation and the answer to part (a) give

$$\frac{\partial \vec{B}}{\partial t} = -c \nabla \times \vec{E} = \nabla \times (\vec{v} \times \vec{B})$$

Busting out our favorite vector triple product identity tells us that

$$\frac{\partial \vec{B}}{\partial t} = (\vec{B} \cdot \nabla)\vec{v} - (\vec{v} \cdot \nabla)\vec{B} + \vec{v}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{v})$$

Flow (not necessarily fluid) incompressibility implies $\nabla \cdot \vec{v} = 0$ and the absence of magnetic monopoles implies that $\nabla \cdot \vec{B} = 0$. Thus, our induction equation simplifies to (2), as desired.

(e) [10 points] Now assume the plasma’s poloidal velocity is zero (as would be the case if the plasma is in hydrostatic equilibrium along a poloidal field line). That is, take $\vec{v}$ to be purely azimuthal:

$$\vec{v} = v(\vec{r})\hat{\phi} = \vec{\Omega}(\vec{r}) \times \vec{r} = [\Omega(\vec{r})\hat{z}] \times \vec{r} = [\Omega(\vec{r})d] \hat{\phi}$$

where $\vec{r}$ is the displacement vector from the origin (i.e., the radius vector in spherical coordinates), and $\vec{\Omega}\hat{z}$ is the local angular velocity of plasma.

At this stage, there is no reason to believe that $\Omega$ cannot vary arbitrarily with position, i.e., the plasma might be differentially rotating in an arbitrary way. In fact, though, part (f) will show that $\Omega$ is not free to vary arbitrarily.

Prove from (2) and (3), the condition of steady state, and the absence of magnetic monopoles that

$$\frac{\partial B_d}{\partial \phi} = \frac{\partial B_z}{\partial \phi} = \frac{\partial B_\phi}{\partial \phi} = 0$$

(4)
That is, the field must be axisymmetric. This makes sense: if the stellar magnetic field were non-axisymmetric (e.g., the star had starspots), then the field seen in the lab frame would be non-steady (it would vary with time at a fixed location as a starspot rotated beneath you).

Hint: I proved \( \partial B_\phi / \partial \phi = 0 \) last, by considering \( \partial (\nabla \cdot \vec{B}) / \partial \phi \).

As the flow is steady, \( \partial \vec{B} / \partial t = 0 \). So our result from (d) becomes

\[
(\vec{B} \cdot \nabla) \vec{v} = (\vec{v} \cdot \nabla) \vec{B}
\]

Let’s look at the components of this vector relation, keeping in mind that \( \vec{v} = v(\vec{r}) \hat{\phi} \). The \( \hat{d} \) component becomes

\[
-\frac{B_\phi v_\phi}{d} = \frac{v_\phi}{d} \frac{\partial B_d}{\partial \phi} - \frac{B_\phi v_\phi}{d}
\]

Or

\[
\frac{v_\phi}{d} \frac{\partial B_d}{\partial \phi} = 0
\]

So, \( B_d \) has no \( \phi \) dependence.

The \( \hat{z} \) component is even simpler. It becomes

\[
0 = \frac{v_\phi}{d} \frac{\partial B_z}{\partial \phi}
\]

So \( B_z \) is also \( \phi \) independent.

Unfortunately, the \( \hat{\phi} \) component will not prove immediately illuminating, though we will return to it later. We know that

\[
\nabla \cdot \vec{B} = 0
\]

Therefore,

\[
\frac{\partial}{\partial \phi} (\nabla \cdot \vec{B}) = 0
\]

Expanding gives us

\[
\frac{\partial}{\partial \phi} \left( \frac{1}{d} \frac{\partial}{\partial d} (dB_d) + \frac{1}{d} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} \frac{\partial B_\phi}{\partial \phi} \right) = 0
\]

Partial derivatives commute so taking the \( \phi \) derivatives all the way through gives us

\[
\frac{1}{d} \frac{\partial}{\partial d} \left( \frac{d B_d}{\partial \phi} \right) + \frac{1}{d} \frac{\partial^2 B_\phi}{\partial \phi^2} + \frac{\partial^2 B_z}{\partial z \partial \phi} = 0
\]

We know that \( B_d \) and \( B_z \) are \( \phi \) independent so this expression immediately simplifies to

\[
\frac{\partial^2 B_\phi}{\partial \phi^2} = 0
\]

Integrating once gives us

\[
\frac{\partial B_\phi}{\partial \phi} = A
\]

\(^2\)This is not the same as \( \partial \vec{B} / \partial \phi = 0 \), since unit vectors in cylindrical coordinates have non-zero \( \phi \)-derivatives.
a constant. So
\[ B_\phi(\phi + 2\pi) = B_\phi(\phi) + 2\pi A \]
But \( B_\phi \) had best be a single valued function. That means
\[ B_\phi(\phi + 2\pi) = B_\phi(\phi) \]
So \( A = 0, \)
\[ \frac{\partial B_\phi}{\partial \phi} = 0 \]
completing the proof.

(f) [5 points] Show that (e) implies \((\vec{B} \cdot \nabla)\vec{\Omega} = 0\). That is, the angular velocity of plasma is constant along a field line. This result is Ferraro’s (1937) Law of Iso-Rotation for axisymmetric fields. All the plasma along a given field line rotates rigidly.

Every field line is rooted on the stellar surface. If the stellar surface is rigidly rotating, then Ferraro’s Law implies that all the plasma within the entire magnetosphere is rigidly rotating with it. The plasma is not moving on Keplerian orbits.

This picture assumes that the plasma does not unduly load the field, i.e., the thermal and ram pressures of the plasma are negligible compared to the magnetic energy densities. This assumption is violated by the Solar wind, more so beyond the Solar wind’s Alfvén point than inside it.

Neutron star magnetospheres are thought to rotate rigidly such that their outer boundaries rotate at nearly the speed of light. Winds from neutron stars can be flung out magnetocentrifugally along open magnetic field lines.

Now we return to the \( \hat{\phi} \) component of our relation
\[ (\vec{B} \cdot \nabla)\hat{v} = (\vec{v} \cdot \nabla)\vec{B} \]
which reads
\[ B_d \frac{\partial v_\phi}{\partial d} + B_\phi \frac{\partial v_\phi}{\partial \phi} + B_z \frac{\partial v_\phi}{\partial z} = v_d \frac{\partial B_\phi}{\partial d} + v_\phi \frac{\partial B_\phi}{\partial \phi} \]
Because only the \( \phi \)-component of \( v \) is non-zero, and because \( \partial B_\phi / \partial \phi = 0 \), this equation becomes
\[ B_d \frac{\partial v_\phi}{\partial d} + B_\phi \frac{\partial v_\phi}{\partial \phi} + B_z \frac{\partial v_\phi}{\partial z} = \frac{v_\phi B_d}{d} \]
Plugging in \( v_\phi = \Omega d \) gives us
\[ B_d \Omega + B_d \frac{\partial \Omega}{\partial d} + B_\phi \frac{\partial \Omega}{\partial \phi} + B_z \frac{\partial \Omega}{\partial z} = B_d \Omega \]
which becomes
\[ \vec{B} \cdot (\nabla \Omega) = 0 \]
But

$$\vec{\Omega} = \Omega(\vec{r})\hat{z}$$

and thus $$(\vec{B} \cdot \nabla)\vec{\Omega}$$ has only a $z$-component (given in the Course Reader) which equals $\vec{B} \cdot (\nabla\Omega) = 0$.

(g) [5 points] Ignore parts (a)–(f) above, and consider a star with a hard surface that is rigidly rotating. The star has a time-steady, axisymmetric, purely poloidal magnetic field that extends through all space. Plasma pervades the field, i.e., plasma fills the space above the star.

In the frame rotating with the star, there is only a static magnetic field and no electric field. In this frame, the charged particles of the plasma surrounding the star execute pure gyro-motion around field lines (assuming infinite conductivity, and neglecting forces like gravity, pressure, Coriolis, and centrifugal forces that are assumed small compared to electromagnetic forces; recall lecture where we made this same “EM beats gravity” approximation in analyzing the equation of motion for electrons).

How does the plasma move in the lab frame? Show that your answer is consistent with $\vec{E} \times \vec{B}$ drift (problem 1).

Note that parts (a)–(f) did NOT assume the field was purely poloidal. So what we are showing here in part (g) is a weak (restrictive) version of Ferraro’s Law of Isorotation. But it is so much easier to see and prove than the more general case.

In the frame rotating with the star at angular frequency $\Omega$, there is only a time-steady $\vec{B}$ and no $\vec{E}$. The charged particles in the plasma just gyrate around the stationary magnetic field lines and don’t have any bulk motion. Now shift back into the lab frame, so that the plasma has velocity $\vec{v} = \Omega d \hat{\phi}$, where $d$ is the cylindrical radius. In other words, the plasma rigidly rotates with the magnetized star. This proves rigid rotation of magnetospheric plasma.

Now we show that this rigid rotation is consistent with $\vec{v}_{\text{drift}} = -c\vec{E} \times \vec{B}/B^2$. In the lab frame, there is an electric field, $\vec{E} = -\vec{v} \times \vec{B}/c$. Because it is assumed — for this part (g) only — that the field is purely poloidal, we have $|\vec{E}| = |vB/c|$, with the direction of $\vec{E}$ purely perpendicular to $\vec{B}$. Then $\vec{v}_{\text{drift}} = c\vec{E} \times \vec{B}/B^2 = +\vec{v} = \Omega d \hat{\phi}$ (you can verify the direction by repeated application of the right-hand rule).

Note that if $\vec{B}$ were not purely poloidal, then $\vec{v}$ and $\vec{B}$ would not be perpendicular, $|\vec{E}| \neq |vB/c|$, and the above $\vec{E} \times \vec{B}$ drift velocity, while still present, would not capture the full rotational speed $\vec{v} = \Omega d \hat{\phi}$. Presumably there are other drift velocities arising from $B_\phi$ that would make up for the shortfall; there must be, to enforce rigid rotation and Ferraro’s law.