Problem 1. MHD Modes

(a) Derive the eigenmodes for Alfvén waves, restricting your attention to the case where the wavevector \( \vec{k} \) is parallel to the background field \( \vec{B}_0 = B_0 \hat{z} = B_0 \hat{3} \). That is, write down the Cartesian components for the perturbation velocity \( \delta \vec{u} = [\delta u_1, \delta u_2, \delta u_3] \) and the perturbation magnetic field \( \delta \vec{B} = [\delta B_1, \delta B_2, \delta B_3] \) when \( \vec{k} \parallel \vec{B}_0 \).

Your solution should be complete up to an overall normalization constant, i.e., the wave amplitude. Express your answers for the eigenmode components in terms of the magnitude of the perturbation velocity \( \delta u = |\delta \vec{u}| \). All six numbers \( \delta u_1, \delta u_2, \delta u_3, \delta B_1, \delta B_2, \delta B_3 \) should be proportional to the free constant \( \delta u \). As part of your answer you may use other variables such as the Alfvén velocity \( u_A = \sqrt{B_0^2/(4\pi \rho_0)} \), the phase velocity \( u_{ph} = \omega/k \), and the sound speed \( c_s \).

Decide also whether the mode is compressive, and whether it is transverse or longitudinal. You do not need to write down the full eigenmode solution for \( \delta \rho \); you need only decide whether \( \delta \rho \) is non-zero.

Optional: draw a picture of the mode.

We start with equations (14.1.25) and (14.1.26) from Sturrock, which follow from \( \vec{k} = (k \sin \theta, 0, k \cos \theta) \); \( \theta = 0 \) corresponds to \( \vec{k} \parallel \vec{B} \) (this part a) and \( \theta = \pi/2 \) corresponds to \( \vec{k} \perp \vec{B} \) (the next part b; and note that there is no loss of generality in taking \( \vec{k} \) to lie in the x-z plane). Sturrock identifies the equation in (14.1.26) containing \( \delta u_2 \) as embodying Alfvén waves; that equation is (14.1.27):

\[
(u_{ph}^2 - u_A^2 \cos^2 \theta) \delta u_2 = 0
\]

which for \( \text{non-zero } \delta u_2 \) demands that

\[
u_{ph} = \pm u_A \cos \theta.
\]

Plug this \( u_{ph} \) into (14.1.26) and see that, for arbitrary \( \theta \), the elements of the matrix are all non-zero except for the center entry (technically \( \theta = 0 \) also gives zero for the upper left entry: \( u_{ph}^2 - u_A^2 = 0 \) for \( \theta = 0 \). We’ll get to this exception later). This implies that \( \delta u_1 = \delta u_3 = 0 \). Plug this result into (14.1.25) and find that \( \delta B_1 = \delta B_3 = 0 \) and \( \delta B_2 = -B_0 \delta u_2 \cos \theta/u_{ph} = \mp B_0 \delta u_2/u_A \). Notice all the boxed results hold for arbitrary \( \theta \)—so the eigenmode as just described for this part (a) is identical to the eigenmode for part (b).

To determine if the modes are compressive, we use Sturrock 14.1.13:

\[-i \omega \delta \rho + i \rho \vec{k} \cdot \delta \vec{u} = 0 \rightarrow \delta \rho = \mp \frac{\rho}{\omega} \vec{k} \cdot \delta \vec{u} \]
For $\vec{k} = k\hat{z}$, this becomes
\[ \delta\rho = \rho \frac{\delta u_3}{u_{ph}} \]

Since $\delta u_3 = 0$ for the Alfvén mode, we have $\delta\rho = 0$.

Putting it all together for $\theta = 0$:

- $u_{ph} = \pm u_A$

\[
\begin{align*}
\delta \vec{u} &= \delta u \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
\delta \vec{B} &= \mp \frac{B_0}{u_A} \delta u \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\end{align*}
\]

$\delta\rho = 0 \rightarrow$ not compressive

Since $\delta \vec{u} \perp \vec{k}$, the Alfvén mode is transverse.

Now for the technical exception for $\theta = 0$. From (14.1.26) we see that if $\theta = 0$, the upper left corner entry is identical to the center entry: both equal zero for Alfvén modes. This just reflects rotational symmetry of the problem for $\theta = 0$: the $\delta u_1 = 0, \delta u_2 \neq 0$ mode is physically equivalent to the $\delta u_1 = 0, \delta u_2 = 0$ that we wrote down above, by rotational symmetry about the $\vec{k} \parallel \vec{B} \parallel \hat{z}$ axis. So we could just as well write down, for this $\vec{k} \parallel \vec{B}$ case,

- $u_{ph} = \pm u_A$

\[
\begin{align*}
\delta \vec{u} &= \delta u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
\delta \vec{B} &= \mp \frac{B_0}{u_A} \delta u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{align*}
\]

$\delta\rho = 0 \rightarrow$ not compressive

Full credit is awarded for this or the previous statement.

The picture of the mode is what we have been drawing in lecture: imagine a harp whose strings are oriented vertically, and perturbed such that they are pulled alternately right and left.

(b) Repeat (a) but for Alfvén waves with $\vec{k}$ perpendicular to $\vec{B}_0$. Do such waves propagate?

As noted above, practically all of our reasoning so far is valid for arbitrary $\theta$. The only place we plugged in for $\theta$ was in evaluating the phase velocity $u_{ph}$; here for $\theta = \pi/2$, we have $u_{ph} = 0$.

Also (3) evaluates to $\delta\rho = +\frac{\rho}{\omega}k_x \hat{x} \cdot \delta u_y \hat{y} = 0$. So we have:
\( u_{\text{ph}} = 0 \): 

\[
\begin{pmatrix}
\delta \mathbf{u} \\
\delta \mathbf{B}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\delta u_x \\
\delta u_z
\end{pmatrix}
\]

\( \delta \rho = 0 \rightarrow \text{not compressive} \)

A zero \( u_{\text{ph}} \) means the wavefronts don’t travel anywhere. The \textbf{transverse} disturbances are purely in \( \hat{y} \) and the wavevector points in \( \hat{x} \) direction. We have vertically oriented sheets alternately sliding into and out of the page as one travels to the right down the \( x \)-axis. We have \( \delta B_y \) varying sinusoidally down the \( x \)-axis, but because adjacent sheets aren’t connected by the same field line, there’s no magnetic tension acting anywhere.\(^1\) So the sheets of matter, though set in motion in \( y \) and with non-zero perturbation fields, don’t snap back: the wavefronts are frozen.

Curiously, however, the group velocity \( u_{g,3} = \partial \omega / \partial k_3 = \partial \omega / \partial (k \cos \theta) = \pm u_A \), apparently irrespective of \( \theta \). I think a non-zero group velocity holds for all \( \theta \) EXCEPT for \( \theta = \pi / 2 \). For any \( \theta \neq \pi / 2 \), a wave will have \( \tilde{B} \) variations in the \( z \) direction. As \( \theta \) approaches but does not reach \( \pi / 2 \), the \( z \)-wavelength will grow super-long but will remain finite. \textit{Any} variation along \( z \) is enough for magnetic tension to send that sinusoidal disturbance up (and down) the field line at the full Alfven velocity. The only case where this doesn’t happen is the “pathological” case when \( \theta = \pi / 2 \) and there is strictly no variation whatsoever along \( z \) and we return to the case where the wavefronts are not connected by field lines and are therefore frozen.

\((c)\) Repeat (a) but for fast and slow magneto-sonic waves having \( \mathbf{k} \) parallel to \( \tilde{B}_0 \). Does any wave \textit{not} propagate?

From Sturrock 14.1.28 we have, for \( \theta = 0 \):

\[
\begin{pmatrix}
\left( u_{\text{ph}}^2 - u_A^2 \right) & 0 \\
0 & \left( u_{\text{ph}}^2 - c_s^2 \right)
\end{pmatrix}
\begin{pmatrix}
\delta u_x \\
\delta u_z
\end{pmatrix} = 0
\]

So we find two solutions: \( \delta u_x \neq 0 \) demands \( u_{\text{ph}} = u_A \), and \( \delta u_z \neq 0 \) demands \( u_{\text{ph}} = c_s \). Meanwhile \( \delta u_y = 0 \) (recall \( \tilde{B}_0, \delta \mathbf{u}, \delta \mathbf{B}, \mathbf{k} \) are in the same plane, which must be the \( x-z \) plane since we have chosen \( \mathbf{k} \) to lie in the \( x \) direction). And from (14.1.25) we have, for \( \theta = 0 \),

\[ \delta B_x = -\frac{B_0}{u_{\text{ph}}} \delta u_x \hat{x}. \]

Finally, from (4),

\[ \delta \rho = +\rho \delta u_3 / u_{\text{ph}}. \]

To summarize:

\(^1\)There are magnetic pressure gradients in this case—but these scale as \( \delta B_y^2 \) and are therefore missed from our linear wave analysis.
• \( u_{ph} = c_s \):

\[
\delta \vec{u} = \delta u \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

\[
\delta \vec{B} = 0
\]

\[
\delta \rho = +\frac{\rho}{c_s} \delta u \rightarrow \text{compressive}
\]

• \( u_{ph} = u_A \):

\[
\delta \vec{u} = \delta u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\delta \vec{B} = -\frac{B_0 \delta u}{u_A} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\delta \rho = 0 \rightarrow \text{not compressive}
\]

Which mode is 'fast' or 'slow' is just determined by which of the two speeds, \( v_A \) and \( c_s \) is larger: the fast mode has \( u_{ph} = \max(v_A, c_s) \) and the slow mode has \( u_{ph} = \min(v_A, c_s) \). Notice that the \( u_{ph} = u_A \) case is just the \( \text{transverse} \) Alfven mode of part (a)—the Alfven mode and one of the magnetosonic modes (either fast or slow) merge to become a single mode when \( \vec{k} \parallel \vec{B}_0 \). Likewise the \( u_{ph} = c_s \) case is just a normal \( \text{longitudinal} \) sound wave.

\( (d) \) Repeat (a) but for fast and slow magneto-sonic waves having \( \vec{k} \) perpendicular to \( \vec{B}_0 \). Does any wave not propagate?

From Sturrock 14.1.28 we have, for \( \theta = \pi/2 \):

\[
\begin{pmatrix} u_{ph}^2 - u_A^2 - c_s^2 \\ 0 \\ u_{ph}^2 \end{pmatrix} \begin{pmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{pmatrix} = 0
\]

So we find two solutions: \( \delta u_x \neq 0 \) demands \( u_{ph}^2 = u_A^2 + c_s^2 \), and \( \delta u_z \neq 0 \) demands \( u_{ph} = 0 \) (which is obviously slow compared to the other solution). Meanwhile \( \delta u_y = 0 \) (recall \( B_0, \delta \vec{u}, \delta \vec{B}, \vec{k} \) are in the same plane, which must be the \( x\)-\( z \) plane since we have chosen \( \vec{k} \) to lie in the \( x \) direction). From (14.1.25) we have \( \delta B_3 = B_0 \delta u_x / u_{ph} \). And from (3) we have, for \( \theta = \pi/2 \), \( \delta \rho = (\rho/\omega)k \delta u_x \).

So the slow and fast magneto-sonic wave eigenmodes for \( \vec{k} \perp \vec{B}_0 \) are:
• slow: \( u_{ph} = 0 \)

\[
\begin{align*}
\delta \mathbf{u} &= \delta u 
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \\
\delta \mathbf{B} &= 0 \\
\delta \rho &= 0 \rightarrow \text{not compressive}
\end{align*}
\]

This technically transverse slow mode doesn’t propagate; it just corresponds to vertical sheets alternately sliding up and down (parallel to \( \mathbf{B}_0 \)) as one travels down the \( x \)-axis, generating no pressure or tension forces.

• fast: \( u_{ph}^2 = u_A^2 + v_s^2 \)

\[
\begin{align*}
\delta \mathbf{u} &= \delta u 
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \\
\delta \mathbf{B} &= \frac{B_0 \delta u}{u_{ph}} 
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \\
\delta \rho &= +\frac{\rho}{u_{ph}} \delta u \rightarrow \text{compressive}
\end{align*}
\]

Notice how \( \delta \rho \) and \( \delta \mathbf{B} \) are in phase with each other: magnetic pressure and thermal pressure work together to make the wave speed larger than either the sound speed or the Alfven speed. Because \( \delta \mathbf{u} \parallel \mathbf{k} \), this is a longitudinal mode.

Problem 2. General Ohm and Three Kinds of Non-Ideal MHD

Consider a fluid that is composed predominantly of neutrals, with a small mixture of ions and electrons. Each ion and electron has the same absolute magnitude of charge \( e (> 0) \). The mass of an ion is \( m_i \), the mass of an electron is \( m_e \), and the number density of ions equals the number density of electrons, \( n_i = n_e \) (net charge neutrality). The fluid is threaded with a magnetic field \( \mathbf{B} \).

Much of this problem was inspired by papers by Wardle (e.g., Wardle 1999; Wardle & Ng 1999; Salmeron & Wardle 2003; Wardle 2007).

The neutrals (each of mass \( m \) with no subscript; e.g., H atoms) provide the bulk of the resistance (a.k.a. collisional drag) against the flow of ions and electrons. A single ion collides with neutrals at a rate \( \nu_i \) (units of per second). A single electron collides with neutrals at a rate \( \nu_e \).\(^2\) The Hall

\(^2\)In the literature (including Shu), people write \( \mu_i \nu_i / m_i = \rho \gamma_i \), where \( \rho \) is the mass density of neutrals and \( \gamma_i = \langle \sigma w \rangle / (m + m_i) \), with the bracketed numerator equal to the collisional rate coefficient (collisional cross section multiplied by relative velocity) averaged over a distribution of relative velocities between ions and neutrals (typically Maxwellian). An expression for \( \gamma_i \) is given in Shu. An analogous statement applies for an electron colliding with neutrals (\( \mu_e \nu_e / m_e = \rho \gamma_e \)).
parameter is the ratio of the particle gyrofrequency (a.k.a. cyclotron frequency) to the collision rate:

\[ \beta_i \equiv \frac{eB}{m_i c \mu_i \nu_i} \quad (5) \]

\[ \beta_e \equiv \frac{eB}{m_e c \mu_e \nu_e} \quad (6) \]

where \( c \) is the speed of light, \( \mu_i \equiv \frac{mm_i}{m + m_i} \) is the “reduced mass” characterizing collisions between ions and neutrals, and \( \mu_e \equiv \frac{mm_e}{m + m_e} \) is the reduced mass characterizing collisions between electrons and neutrals. For a pure hydrogen plasma, \( \mu_i = m_i/2 \) and \( \mu_e \approx m_e \).

A particle with \( \beta \gg 1 \) is said to be “tied to the field”: it can gyrate many times around a field line before it gets perturbed by a neutral. Generally \( \beta_e \gg \beta_i \) —although not by the full factor of \( m_i/m_e \sim 42^2 \), since there are mass dependences in \( \nu_i \) and \( \nu_e \) (i.e., electrons have random thermal velocities that are larger than ion thermal velocities which makes \( \nu_e/\nu_i > 1 \)).

This problem explores how the electrical conductivity of the fluid changes in the limits of high and low \( \beta \), and by extension how the induction equation for \( \partial \vec{B}/\partial t \) changes. We will recover Ohmic diffusion as discussed in class, but also discover new non-ideal regimes.

Many parts of this problem can be done independently of one another. For example, one does not need to solve (a) (which is a bit involved) to solve other parts. Also note that each part may ask several questions.

(a) In class we stated that in many situations the equation of motion of a charged particle is dominated by electromagnetic forces and collisional drag. These forces typically overwhelm gravity and inertial forces (whether this is true or not in any given situation can be tested to order-of-magnitude).

In the frame co-moving with the neutrals,

\[ +e(\vec{E}' + \vec{u}_i \times \vec{B}/c) - \nu_i \mu_i \vec{u}_i = 0 \quad (7) \]

\[ -e(\vec{E}' + \vec{u}_e \times \vec{B}/c) - \nu_e \mu_e \vec{u}_e = 0 \quad (8) \]

where \( \vec{u}_i \) and \( \vec{u}_e \) are the (bulk) velocities of ions and electrons relative to the neutrals, and the reduced masses \( \mu_i = mm_i/(m + m_i) \) and \( \mu_e = mm_e/(m + m_e) \). The electric field \( \vec{E}' \) is primed to remind us that this is the field seen in the rest frame of the (overwhelmingly) neutral fluid; later we will switch back to the unprimed lab frame. We don’t prime the magnetic field \( \vec{B} \) to remind us that the magnetic field does not change between frames—non-relativistically!

Use the ion and electron equations of motion to derive a generalized Ohm’s Law connecting the current density \( \vec{j} = n_i e \vec{u}_i - n_e e \vec{u}_e \), the components of the electric field that are parallel (\( \vec{E}'_\parallel \)) and perpendicular (\( \vec{E}'_\perp \)) to the magnetic field, and \( \vec{B} \) (the unit vector pointing in the direction of the magnetic field):

\[ \vec{j} = \sigma_\parallel \vec{E}'_\parallel + \sigma_H \vec{B} \times \vec{E}'_\perp + \sigma_P \vec{E}'_\perp \quad (9) \]
where the electrical conductivity parallel to the \( \vec{B} \) field is
\[
\sigma_\parallel = \frac{e c}{B} (n_i \beta_i + n_e \beta_e) ; \tag{10}
\]
the Hall conductivity
\[
\sigma_H = -\frac{e c}{B} \left( \frac{n_i \beta_i^2}{1 + \beta_i^2} - \frac{n_e \beta_e^2}{1 + \beta_e^2} \right) = \frac{e c}{B} \left( \frac{n_i}{1 + \beta_i^2} - \frac{n_e}{1 + \beta_e^2} \right) ; \tag{11}
\]
and the Pedersen conductivity
\[
\sigma_P = \frac{e c}{B} \left( \frac{n_i \beta_i}{1 + \beta_i^2} + \frac{n_e \beta_e}{1 + \beta_e^2} \right) . \tag{12}
\]
The second equality of equation (11) is derived from the first equality using charge neutrality \( n_i = n_e \).

**Hint:** I was able to simplify the algebra by taking, without loss of generality, \( \vec{B} = [0,0,B] \) and \( \vec{E}' = [E'_1,0,E'_||] \). Then use equation (7) to solve for \( \vec{u}_i = [u_{i1}, u_{i2}, u_{i3}] \), likewise equation (8) for the electrons, and use these to find \( \vec{j} \). Finally cast your answer in the coordinate-independent language of equation (9) by recalling your coordinate choices for \( \parallel \) and \( \perp \).

Follow the hint and write out all the components of the ion equation of motion (we’ll skip the \( i \) subscripts for now and restore them later):
\[
e E'_1 + (e/c)u_2 B = \nu_i \mu_i u_1 \tag{13}
\]
\[
- (e/c)u_1 B = \nu_i \mu_i u_2 \tag{14}
\]
\[
e E'_|| = \nu_i \mu_i u_3 \tag{15}
\]
The \( u_3 \) equation decouples from the others so let’s solve for \( u_3 \) first and use it to solve for the field-aligned current (the first term in equation (9)):
\[
u_{i3} = (e/\nu_i \mu_i) E'_|| \tag{16}
\]
\[
u_{e3} = -(e/\nu_e \mu_e) E'_|| \tag{17}
\]
\[
j_3 = e(n_i u_{i3} - n_e u_{e3}) \tag{18}
\]
\[
e^2 \left( \frac{n_i}{\nu_i \mu_i} + \frac{n_e}{\nu_e \mu_e} \right) E'_|| \tag{19}
\]
\[
= \frac{e c}{B} \left( \frac{n_e B^2}{c \nu_e \mu_e} + \frac{n_e B^2}{c \nu_e \mu_e} \right) E'_|| \tag{20}
\]
\[
= \sigma E'_|| \tag{21}
\]
OK 1/3 done. Now go back and solve the two other equations for \( u_1 \) and \( u_2 \). First \( u_1 \):
\[
u_{i1} = \frac{e E'_1}{\nu_i \mu_i \left( 1 + e^2 B^2 / c^2 \nu_i^2 \mu_i^2 \right)} \tag{22}
\]
\[
= \frac{e c}{B} \frac{n_i \beta_i}{1 + \beta_i^2} \tag{23}
\]
from which it follows that \( j_1 = \sigma_P E'_\perp \). That’s 2/3 done. Next \( u_2 \):

\[
\begin{align*}
  u_{i2} &= -eB \
  \frac{eE'_\perp}{ev_i\mu_i \nu_i\mu_i(1+\beta_i^2)} \\
  u_{e2} &= +eB \
  \frac{-eE'_\perp}{ev_e\mu_e \nu_e\mu_e(1+\beta_e^2)}
\end{align*}
\]  

(24)

(25)

Notice how \( u_{i2} \) and \( u_{i3} \) have the same sign, because the charge comes in squared. This recalls how \( E \times B \) drift is in the same direction for particles regardless of the sign of their charge. Proceeding:

\[
\begin{align*}
  n_i e u_{i2} &= \frac{-ec n_i \beta_i^2}{B(1+\beta_i^2)} \\
  -n_e e u_{i3} &= \frac{+ec n_i \beta_i^2}{B(1+\beta_i^2)}
\end{align*}
\]  

(26)

(27)

from which it follows that \( j_2 = \sigma_H \hat{B} \times \vec{E}'_\perp \), after realizing that \( \hat{3} \times \hat{1} = \hat{2} \). Now we’re 3/3 done.

(b) The low-\( \beta \) limit: suppose \( \beta_i \ll \beta_e \ll 1 \) (charged particles are not well-tied to the field because collisions dominate).

First decide which cross-field current is more important. Which is larger, the second or the third term on the RHS of equation (9)? Drop the smaller term.

Taylor expanding we have \( \sigma_H \propto 1 - \beta_i^2 - 1 + \beta_e^2 \simeq \beta_e^2 \), while \( \sigma_P \propto \beta_i(1-\beta_i^2) + \beta_e(1-\beta_e^2) \simeq \beta_e \).

So \( \sigma_P > \sigma_H \) and can drop the \( \hat{B} \times \vec{E}'_\perp \) term.

Examine the relative magnitudes of the conductivities of the two terms that you have left, working in the low \( \beta_i \ll \beta_e \ll 1 \) limit. Simplify accordingly as much as you can, and solve for \( \vec{E}' \) in terms of \( \vec{j} \).

We have \( \sigma_P \simeq \sigma_\parallel \simeq (ec/B)n_e\beta_e \). Define \( \sigma = (ec/B)n_e\beta_e \) (compare to the expression in class). Then \( \vec{j} = \sigma \vec{E}' \).

In the lab frame, the velocity of the neutrals is \( \vec{u} \).

Write down \( \vec{E} \), the electric field seen in the lab frame, in terms of \( \vec{E}' \), \( \vec{u} \), and other given quantities.

This is a Lorentz transform by \( -\vec{u} \):  

\[ \vec{E} = \vec{E}' - \vec{u} \times \vec{B}/c \]

Plug your expression for \( \vec{E} \) into Maxwell’s induction equation:

\[
\frac{\partial \vec{B}}{\partial t} = -e\nabla \times \vec{E}.
\]  

(28)

\(^3\)This is slightly misleading notation, since we have been using \( \vec{u}_i \) and \( \vec{u}_e \) to denote velocities relative to the neutrals.
Replace $\vec{j}$ with Ampere’s law:
\[ \nabla \times \vec{B} = 4\pi \vec{j}/c \quad (29) \]
and use the vector identity
\[ \nabla \times (\nabla \times \vec{B}) = \nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B} \quad (30) \]
and the fact that there are no magnetic monopoles (in this universe)
\[ \nabla \cdot \vec{B} = 0 \quad (31) \]
to write down a formula for $\partial \vec{B}/\partial t$ in terms of $\vec{u}$ and $\vec{B}$ and whatever conductivities and constants you need.

\[ \frac{\partial \vec{B}}{\partial t} = \frac{c^2}{4\pi \sigma} \nabla^2 \vec{B} + \nabla \times (\vec{u} \times \vec{B}) \]

If all is well, the result will look familiar—this low $\beta$ limit is the regime of Ohmic diffusion that we encountered in class. Is Ohmic diffusion important in high density or low density environments?

[High density] in two senses. First, to drive down $\beta \propto 1/\nu \propto 1/n$. Second, to drive down the fractional ionization $f_e = n_e/n$, since the Ohmic diffusion terms scales as $1/\sigma \propto n/n_e \propto 1/f_e$. Increasing the overall density tends to decrease $n_e/n$; this is true not just for thermal ionization (Saha equation predicts fractional ionization to decrease with increasing density) but also for non-thermal ionization processes like photo-ionizations or cosmic-ray ionizations. In all of these cases, the rate at which ions and electrons recombine (reducing the fractional ionization) scales with density to a power greater than that characterizing ionizations (e.g., in photoionization equilibrium, the radiative recombination rate between electrons and ions goes as $n^2$, steeper than the photoionization rate which goes as $n$; and in thermal ionization, the rate of three-body recombinations goes as $n^3$, again steeper than the rate of two-body collisional ionizations which goes as $n^2$).

(c) The high $\beta$ limit: now assume $\beta_e \gg \beta_i \gg 1$ (particles are well-tied to the field).

As in part (b), first decide which cross-field current is more important. Which is larger, the second or the third term on the RHS of equation (9)? Drop the smaller term.

We have $\sigma_H \propto 1/\beta_i^2$ and $\sigma_P \propto 1/\beta_i$ so $\sigma_P > \sigma_H$ and we can drop the $\vec{B} \times \vec{E}'_{\perp}$ term.

Now try to decide which is larger, $E'_{\perp}$ or $E'_{||}$. Assume that $j_{\perp} \sim j_{||}$ and compare the relative magnitudes of the conductivities in equation (9). Drop the term that has the smaller electric-field component, and solve for the dominant electric-field component.

We have $\sigma_{||} \gg \sigma_P$ (much easier to slide along the field line than to cross it) so $E'_{||} \gg E'_{\perp}$ (assuming $j_{||} \sim j_{\perp}$). So we have, approximately, $\vec{E}' \approx \vec{E}'_{\perp} \approx \vec{j}/\sigma_P$.

Find that:
\[ \vec{E}' = -\frac{(\vec{j} \times \vec{B}) \times \vec{B}}{c^2 \nu_i n_i \mu_i} \quad (32) \]
Recall we simplified the algebra by taking \( \perp \) to be in the \( \hat{1} \) direction. So from the boxed result above, we have \( \vec{j} \approx j \hat{1} \). Cross this into \( \vec{B} = B \hat{3} \) to get \( -jB^2 \hat{2} \). Then cross again with \( \vec{B} \) to get \( -jB^2 \hat{1} \).

So we have

\[
\vec{E}' = \frac{- (\vec{j} \times \vec{B}) \times \vec{B}}{B^2 \sigma P} = \frac{- (\vec{j} \times \vec{B}) \times \vec{B}}{B^2 (ec/B)n_i/\beta_i},
\]

\[
= \frac{- (\vec{j} \times \vec{B}) \times \vec{B}}{c^2 n_i \mu_i \nu_i}
\]

as requested.

As in (b), Lorentz transform back to the lab frame to find \( \vec{E} \), replace \( \vec{j} \) with \( \vec{B} \) using Ampere’s law, and insert into the induction equation to write down a formula for \( \partial \vec{B}/\partial t \) in terms of \( \vec{u} \) and \( \vec{B} \) (you may compare your answer to equation 27.12 of Shu, keeping in mind that our \( \nu_i \mu_i / m_i \) is his \( \rho \gamma_i \), and our \( n_i m_i \) is his \( \rho_i \)).

This procedure will yield the same \( \nabla \times (\vec{u} \times \vec{B}) \) field-freezing term as in (b), plus a new non-ideal term that looks like:

\[
-c \nabla \times \vec{E}' = -c \nabla \times \left[ \frac{- (\vec{j} \times \vec{B}) \times \vec{B}}{c^2 n_i \mu_i} \right]
\]

\[
= +c \nabla \times \left[ \frac{((\nabla \times \vec{B}) \times \vec{B}) \times \vec{B}}{c^2 n_i \mu_i} \right] \frac{c}{4\pi}
\]

\[
= \nabla \times \left[ \frac{\vec{B} \times (\vec{B} \times (\nabla \times \vec{B}))}{4\pi n_i \mu_i} \right]
\]

which matches Shu 27.12 once we replace \( \nu_i \mu_i \) with \( \rho \gamma_i m_i \) and \( n_i m_i = \rho_i \).

This is the induction equation with a new non-ideal term: that for ambipolar diffusion. Insofar as it involves two spatial derivatives, this non-ideal term also acts to diffuse away magnetic field, like Ohmic diffusion. But unlike Ohmic diffusion, the diffusivity depends on the quantity being diffused; the diffusivity scales as \( B^2 \) (see also Shu page 364). So roughly speaking, where \( B \) is stronger, the field diffuses away faster.

The similarity with the Ohmic diffusion term can be made even more clear by dispensing with the double cross products and keeping \( \vec{E}' = \vec{j} / \sigma P \). Inserting this into the induction equation and replacing \( \vec{j} \) with Ampere’s Law then gives \( (c^2/4\pi \sigma P) \nabla^2 \vec{B} \), which looks just like the Ohmic diffusion term with \( \sigma \) replaced by \( \sigma P \propto 1/B^2 \); the magnetic diffusivity increases with the magnetic field strength. (It is not clear to me why people don’t write it like this.)

(d) In ambipolar diffusion, the Hall parameters \( \beta_e \) and \( \beta_i \) are both \( \gg 1 \), which means both electrons and ions are well-tied to the field. So when the field diffuses away, it carries both electrons and ions
with it—hence the term “ambipolar,” meaning of both polarities. The magnetized charged plasma leaves the neutrals behind. Ambipolar diffusion is responsible for, e.g., star formation (enables gas to slip out of field).

We can estimate the relative ion-neutral slip speed—recall we called this \( u_i \) in part (a)—as follows. Return to the ion and electron equations of motion (7) and (8) to derive:

\[
\frac{\mathbf{j} \times \mathbf{B}}{c} = n_i \nu_i \mu_i \mathbf{u}_i
\]

(39)

where we have dropped the electron drag term which is small compared to the ion drag term (mostly because \( \mu_i \gg \mu_e \), the fact that \( \nu_e > \nu_i \) notwithstanding; also \( u_i \sim u_e \) because both are well-tied to the field, notwithstanding the usual tiny ion-electron slip velocity that generates current).

Multiply equations (7) and (8) by \( n_i (= n_e) \) and add to obtain the equation above, dropping the electron drag term as instructed.

Pull out Ampere’s law again to find that, to order-of-magnitude,

\[
u_i \sim \frac{B^2}{4\pi n_i \mu_i \nu_i t} \quad (40)\]

There are astrophysical systems where this drift velocity can be large, e.g., comparable to sound speeds (e.g., Wang & Goodman 2017).

Replace \( \mathbf{j} \times \mathbf{B} \) with \( (c/4\pi)(\nabla \times \mathbf{B}) \times \mathbf{B} \) with \( (c/4\pi)B^2/L \) to find the desired order-of-magnitude relation.

(e) By analogy to how we defined the dimensionless magnetic Reynolds number \( Re_M \), define another dimensionless number, \( Am \), that compares the magnitude of the flux-freezing term to the ambipolar diffusion term in the induction equation that you solved for in part (c). Show that to order-of-magnitude,

\[
Am = \frac{u^2 t}{v_A t_{ni}} \quad (41)
\]

where the Alfven speed \( v_A = \sqrt{B^2/4\pi \rho} \) (where \( \rho \) is the density of neutrals), \( t = L/u \) is a characteristic timescale in the flow, and \( t_{ni} \) is the timescale for a single neutral to collide with an ion in a sea of ions. Thus conclude that if \( u \sim v_A \) (true if ram pressures \( \rho u^2 \) are comparable to magnetic pressures \( B^2 \)), then \( Am \ll 1 \) (flux freezing fails and ambipolar diffusion is important) if \( t_{ni} \gg t \). This makes sense—if a neutral can’t find an ion to collide and share momentum with before the flow changes, then the neutral gets left behind (and notice how we are now talking about a given neutral in a sea of ions, not a given ion in a sea of neutrals—the latter is controlled by the ion Hall parameter).

Take the magnitude of the flux-freezing term \( \nabla \times (\mathbf{u} \times \mathbf{B}) \sim uB/L \) and divide by the magnitude of the ambipolar diffusive term (38) \( \sim B^3/(L^2 4\pi \nu_i n_i \mu_i) \):
\[
Am \sim \frac{uLA\pi n_i \mu_i}{B^2} \tag{42}
\]
\[
\sim \frac{uLA\pi \rho n_i \mu_i}{B^2 \rho} \tag{43}
\]
\[
\sim \frac{u^2 L \nu_i n_i \mu_i}{uu^2 A \rho} \tag{44}
\]
\[
\sim \frac{u^2 t \nu_i n_i \mu_i}{v^2 A \rho} \tag{45}
\]
\[
\sim \frac{u^2 t \nu_i n_i}{v^2 A \rho} \tag{46}
\]
\[
\sim \frac{u^2 t \nu_i n_i}{v^2 A n} \tag{47}
\]
\[
\sim \frac{u^2 t \nu_i n_i}{v^2 A n} \tag{48}
\]
\[
\sim \frac{u^2 t \nu_i n_i}{v^2 A} \tag{49}
\]
\[
\sim \frac{u^2 t}{v^2 A t_{ni}} \tag{50}
\]
\[
\sim \frac{u^2 t}{v^2 A t_{ni}} \tag{51}
\]

Bottom line: is ambipolar diffusion significant in low density or high density environments?

Low density to make the ion-neutral collision time \( t_{ni} \) super-long.

(f) There is a third non-ideal regime. What is true about \( \beta_e \) and \( \beta_i \) here? This is called the Hall diffusion regime.

There is only stone left unturned: when \( \beta_e \gg 1 \) and \( \beta_i \ll 1 \).