Astrophysical Fluid Dynamics – Problem Set 9

Readings: Sturrock 14.1 (Course Reader) on MHD waves; Shu Chapter 22 on MHD waves; Weber & Davis (1967) on the angular momentum of the magnetized solar wind, and Blandford & Payne (1982) on magneto-centrifugal disk winds (Course Reader)

Problem 1. MHD Modes

The answers to this problem are all contained in Sturrock 14.1 (reprinted in the Course Reader) and Shu Chapter 22. You may use as much of Sturrock (or Shu) as you wish; I like deriving Sturrock's results before using them, but however much you want to derive is up to you.

Consider a uniform, adiabatic, perfectly conducting medium of density ρ_0, pressure P_0, and adiabatic index γ, threaded with a magnetic field $\vec{B}_0 = B_0\hat{z} = B_0\hat{3}$. Perturb this medium with a small-amplitude (linear) wave of wavevector $\vec{k}$ and frequency ω.

(a) [7 points] Write down the eigenmodes for Alfven waves, restricting your attention to the case where the wavevector $\vec{k}$ is parallel to $\vec{B}_0$. That is, write down the Cartesian components for the perturbation velocity $\delta\vec{u} = [\delta u_1, \delta u_2, \delta u_3]$ and the perturbation magnetic field $\delta\vec{B} = [\delta B_1, \delta B_2, \delta B_3]$ when $\vec{k} \parallel \vec{B}_0$. Also write down the perturbation density $\delta\rho$ (a scalar) in terms of the magnitude of the perturbation velocity $\delta u \equiv |\delta\vec{u}|$.

Your solution should be complete up to an overall normalization constant, i.e., the wave amplitude. Express your answers for the eigenmode components in terms of the magnitude of the perturbation velocity δu. That is, all seven numbers $\delta u_1, \delta u_2, \delta u_3, \delta B_1, \delta B_2, \delta B_3, \delta\rho$ should be proportional to the free constant δu. Be careful of signs—these are important because they indicate phase relationships between the perturbations.

Express your answers in terms of the Alfven velocity $u_A = \sqrt{B_0^2/4\pi\rho_0}$, the phase velocity $u_{\rm ph} = \omega/|\vec{k}|$, the sound speed $c_s \equiv \sqrt{\gamma P_0/\rho_0}$, and the background density ρ_0.

Specify whether the mode is compressive or not, and whether it is transverse or longitudinal.

To gain physical intuition, you may find it helpful draw a picture of the mode, but this is not required.

Hint: I like Sturrock (14.1.27). Equation (14.1.30) is OK except the units are incorrect.

(b) [7 points] Repeat (a) but for Alfven waves with $\vec{k}$ perpendicular to $\vec{B}_0$. Do such waves propagate? Why or why not?

Note: Sturrock claims $\delta\vec{B} = 0$ for this case. But this is not necessarily what one would conclude from his equation 14.1.25, since $\cos\theta$ (θ being the angle between $\vec{k}$ and $\vec{B}_0$) and the phase velocity (which he calls v_ϕ and we call $u_{\rm ph}$) both go to zero as $\theta \to 90°$. Full credit will be given whether you believe in Sturrock's statement or whether, like me, you believe that $\delta\vec{B}$ should vary smoothly with θ. The choice does not make any physical difference as you will see when you consider the mode's propagation behavior.

(c) [7 points] Repeat (a) but for fast and slow magneto-sonic waves having $\vec{k}$ parallel to $\vec{B}_0$. Does any wave not propagate?

Hint: I like Sturrock (14.1.28) but I don't like (14.1.33) which has bad units and only captures 1 of the 2 magneto-sonic modes.

(d) [7 points] Repeat (a) but for fast and slow magneto-sonic waves having $\vec{k}$ perpendicular to $\vec{B}_0$. Does any wave not propagate?

Problem 2. The Parker Spiral

This problem concerns the steady, axisymmetric, rotating stellar wind from a split magnetic monopole (Weber & Davis 1967). We introduced this system in class; the notation below is the same as that in class (we work in spherical coordinates where r is radius and ϕ is the azimuthal angle).

(a) [2 points] Define the radial Alfvén Mach number to be $M_A(r) \equiv u_r/u_{\rm A}$, where $u_r(r)$ is the radial velocity and $u_{\rm A}(r) = \sqrt{B_r^2/(4\pi\rho)}$ is the radial Alfvén speed for radial field B_r and density ρ. Call the special radius where $M_A = 1$ the Alfvén radius $r = r_A^\star$ (all starred quantities below are evaluated at the Alfvén point).

Using mass conservation $\rho u_r r^2 =$ constant, and $B_r r^2 =$ constant for a split magnetic monopole, show that

$$\frac{M_A^2}{1} = \frac{\rho_A^\star}{\rho} \tag{1}$$

where $\rho_A^\star$ equals the density at $r_A^\star$.

(b) [5 points] In class we showed that

$$u_\phi = \Omega_\odot r \, \frac{M_A^2 \ell/(\Omega_\odot r^2) - 1}{M_A^2 - 1} \tag{2}$$

for constant total (mechanical + magnetic) specific angular momentum $\ell = \Omega_\odot (r_A^\star)^2$.

Insert (a) into the above relations to show that

$$u_\phi = \Omega_\odot r \, \frac{1 - u_r/u_r^\star}{1 - M_A^2} \tag{3}$$

where $u_r^\star$ is evaluated at the Alfvén radius.

Thus show that at $r \ll r_A^\star$, $u_\phi \sim \Omega_\odot r$. That is, inside the Alfvén radius, the wind rotates rigidly with the star. This result follows from Ferraro's law of isorotation for magnetospheres. The magnetosphere centrifugally "flings" fluid parcels with a moment arm whose length is of order $r_A^\star$. This statement applies to u_ϕ only, not to u_r; recall that for low-mass main-sequence stars, the wind is driven radially by gas pressure, not by magnetic fields.

At $r \gg r_A^\star$, u_r continues to grow, but not too much — recall an earlier problem set where we saw that for the isothermal Parker wind, the radial velocity grows only logarithmically with distance beyond the sonic point (technically the slow magneto-sonic point). Assume that at $r \gg r_A^\star$, $u_r > u_r^\star$ by an order-unity factor. Then show that $u_\phi \propto 1/r$ at large distances from the star, as expected for a fluid parcel conserving its mechanical angular momentum.

To summarize, a fluid parcel is centrifugally accelerated by the star's magnetic field inside the Alfvén radius (i.e., inside the star's magnetosphere) where the field lines can be thought of as rigid wires. The field lines lose their rigidity (they get "floppy") outside the Alfvén radius. After the fluid parcel "flies off the handle" at $r_A^\star$, it is no longer magneto-centrifugally accelerated; its rotational velocity u_ϕ then decreases according to the conservation of mechanical angular momentum ru_ϕ.

(c) [10 points] In class we showed

$$r(u_r B_\phi - u_\phi B_r) = -\Omega_\odot B_r r^2 \tag{4}$$

from which it follows that

$$\frac{B_\phi}{B_r} = \frac{u_\phi - \Omega_\odot r}{u_r} \, . \tag{5}$$

Show from (5) and parts (a) and (b) that

$$\frac{B_\phi}{B_r} = -\frac{\Omega_\odot r}{u_r^\star} \frac{1 - r^2/(r_A^\star)^2}{1 - M_A^2} \, . \tag{6}$$

Using arguments similar to those in (b), show that $B_\phi/B_r \propto -r$ in both the $r \ll r_A^\star$ and $r \gg r_A^\star$ limits.

In general, given a vector magnetic field $\vec{B} = B_r \hat{r} + B_\phi \hat{\phi}$, the equation for a field line is $dr/(rd\phi) = B_r/B_\phi$ (this just says that a field line is everywhere tangent to the field). Show that if $B_\phi/B_r = -r/a$ for some positive constant a, a field line traces a trailing Archimedean spiral which by definition obeys $r = -a\phi$. Sketch qualitatively a field line (no need to be quantitatively accurate). At small $r \ll a$, is the field mostly radial or toroidal? At large $r \gg a$, is the field mostly radial or toroidal?

Problem 3. OPTIONAL Magneto-Centrifugal Flinging

[10 points] This problem is based on the classic paper on disk-driven outflows by Blandford & Payne (1982, reprinted in the Course Reader).

Consider a magnetized disk orbiting a central massive object. The central mass dominates the gravitational potential.

Take the magnetic field to be axisymmetric and purely poloidal. Examine a magnetic field line rooted to the disk midplane at distance a from the central object. Above the disk, the field line is straight but tilted away from the star at an angle θ measured from the vertical (disk normal). The field line is mirror-symmetric about the disk plane (pointing toward the disk below the midplane and pointing away from the disk above the midplane). Assume a given field line acts as a rigid wire, co-rotating with the disk where it is rooted (recall Ferraro's law of iso-rotation).

For what angles θ will disk plasma, starting at the disk midplane, slide unstably along a field line, away from the star, thereby producing a magneto-centrifugal outflow? Assume a fluid parcel can travel only parallel to the field (again recall Ferraro's law which states that the poloidal fluid velocity is parallel to the poloidal magnetic field). Assume that the pressure gradient ∇P parallel to the field is negligible (this is a plasma $\beta \ll 1$ problem).

This looks like an MHD fluids problem, but with the given simplifications, it reduces to a bead-on-a-wire mechanics problem. Remember that there are two kinds of equilibrium: stable and unstable.