Problem 1. Generalized Ohm’s Law (Three Kinds of Non-Ideal MHD)

Consider a fluid that is composed predominantly of neutrals, with a small mixture of ions and electrons (cold star-forming molecular clouds and planet-forming disks are examples of such lightly ionized gases). Each ion and electron has the same absolute magnitude of charge $e (> 0)$. The mass of an ion is $m_i$, the mass of an electron is $m_e$, and the number density of ions equals the number density of electrons, $n_i = n_e$ (net charge neutrality). The fluid is threaded with a magnetic field $\vec{B}$. Much of this problem was inspired by papers by Wardle (e.g., Wardle 1999; Wardle & Ng 1999; Salmeron & Wardle 2003; Wardle 2007) which you are free to look up; Wardle 2007 is in the Course Reader.

The neutrals (each of mass $m$ with no subscript; e.g., H atoms, or H$_2$ molecules) provide the bulk of the resistance (a.k.a. collisional drag) against the flow of ions and electrons. That is, collisions between ions and electrons (considered in class) are negligible here compared to collisions between ions and neutrals, and between electrons and neutrals.

In class, when discussing the collisional drag force between two fluids, we considered the combination $\mu \vec{u}_{rel} \nu$ [units of momentum per time = force], where $\nu$ was the collision rate and $\mu \vec{u}_{rel}$ was the momentum exchanged per collision, with $\mu$ equal to the reduced mass between colliding partners, and $\vec{u}_{rel}$ equal to the mean relative velocity. For ions colliding with neutrals, we have $\mu_i \vec{u}_i \nu_i$, where $\mu_i = m_i m_{\text{ne}} / (m + m_i)$, $\vec{u}_i$ is the mean RELATIVE velocity between ions and neutrals, and $\nu_i$ is the collision rate of a single ion in a sea of neutrals.

The literature on lightly ionized fluids (including Shu) uses different notation, which we will also use here. For ions colliding with neutrals, the literature replaces our $\mu_i \nu_i \vec{u}_i$ with the equivalent $\rho m_i \gamma_i \vec{u}_i$, where $\rho$ is the mass density of neutrals and $\gamma_i$ is a collisional rate coefficient. An expression for $\gamma_i$ is given in Shu (page 362; $\gamma_i$ is equally good for ions colliding with neutrals and neutrals colliding with ions). The new notation is purely a matter of convention. The advantage of this notation is that it gives a simple and sensible expression for the mean free time for a given ion to collide in a sea of neutrals (i.e., the time for the ion fluid to lose its mean momentum relative to neutrals): $t_i = \text{[momentum]} / \text{[force]} = m_i \vec{u}_i / (\rho m_i \gamma_i \vec{u}_i) = 1 / (\rho \gamma_i)$. The corresponding single-ion collision frequency is just $1 / t_i = \rho \gamma_i$. Analogous statements apply for an electron colliding with a sea of neutrals (just replace the subscript i with e).

1Sorry, our notation for $\vec{u}_i$ here differs from that in class. In class, $\vec{u}_i$ was the velocity of ions measured in the lab frame. Here $\vec{u}_i$ is the velocity of ions measured RELATIVE TO THE NEUTRAL FLUID. I didn’t want to introduce primes for $\vec{u}_i$ even though it is a relative velocity because if we did, we would wind up writing a TON of primes everywhere (as you’ll see when you do the problem).
The Hall parameter is the ratio of the particle gyrofrequency (a.k.a. cyclotron frequency) to the single-particle collision frequency:

\[ \beta_i \equiv \frac{eB}{m_i c \rho \gamma_i} \]  \hspace{1cm} (1)

\[ \beta_e \equiv \frac{eB}{m_e c \rho \gamma_e} \]  \hspace{1cm} (2)

where \( c \) is the speed of light and \( B \) is the magnitude of the magnetic field. A particle with \( \beta \gg 1 \) is said to be “tied to the field”: it can gyrate many times around a field line before it gets perturbed by a neutral. Generally \( \beta_e \gg \beta_i \) —although not by the full factor of \( m_i/m_e \sim 42^2 \), since there are mass dependences in \( \gamma_i \) and \( \gamma_e \) (i.e., electrons have random thermal velocities that are larger than ion thermal velocities which makes \( \gamma_e/\gamma_i > 1 \)).

This problem explores how the electrical conductivity of the fluid changes in the limits of high and low \( \beta \), and by extension how the induction equation for \( \partial \vec{B}/\partial t \) changes.

Many parts of this problem can be done independently of one another. For example, one does not need to solve (a) (which is a bit involved) to solve other parts. Also note that each part may ask several questions.

(a) [10 points] In class we stated that in many situations the equation of motion of a charged particle is dominated by electromagnetic forces and collisional drag. These forces typically overwhelm gravity, pressure gradients, and inertial forces (whether this is true or not in any given situation can be tested to order-of-magnitude).

In the frame co-moving with the neutrals,

\[ +e(\vec{E}' + \vec{u}_i \times \vec{B}/c) - \rho \gamma_i m_i \vec{u}_i = 0 \]  \hspace{1cm} (3)

\[ -e(\vec{E}' + \vec{u}_e \times \vec{B}/c) - \rho \gamma_e m_e \vec{u}_e = 0 \]  \hspace{1cm} (4)

where \( \vec{u}_i \) and \( \vec{u}_e \) are the (bulk) velocities of ions and electrons RELATIVE TO THE NEUTRALS (see previous footnote). The electric field \( \vec{E}' \) is primed to remind us that this is the field seen in the rest frame of the (overwhelmingly) neutral fluid; later we will switch back to the unprimed lab frame. We WON’T prime the magnetic field \( \vec{B} \) to remind us that the magnetic field does NOT change between frames—assuming non-relativistic speeds.

Use the ion and electron equations of motion to derive a “generalized Ohm’s Law” connecting the current density \( \vec{j} = n_i e \vec{u}_i - n_e e \vec{u}_e \), the components of the electric field that are parallel (\( \vec{E}'_\parallel \)) and perpendicular (\( \vec{E}'_\perp \)) to the magnetic field, and \( \vec{B} \) (the unit vector pointing in the direction of the magnetic field):

\[ \vec{j} = \sigma_\parallel \vec{E}'_\parallel + \sigma_H \vec{B} \times \vec{E}'_\perp + \sigma_P \vec{E}'_\perp \]  \hspace{1cm} (5)

where the electrical conductivity parallel to the \( \vec{B} \) field is

\[ \sigma_\parallel = \frac{ec}{B} (n_i \beta_i + n_e \beta_e) \]  \hspace{1cm} (6)
the Hall conductivity
\[
\sigma_H = -\frac{ec}{B} \left( \frac{n_i\beta_i^2}{1 + \beta_i^2} - \frac{n_e\beta_e^2}{1 + \beta_e^2} \right) = \frac{ec}{B} \left( \frac{n_i}{1 + \beta_i^2} - \frac{n_e}{1 + \beta_e^2} \right);
\]
and the Pedersen conductivity
\[
\sigma_P = \frac{ec}{B} \left( \frac{n_i\beta_i}{1 + \beta_i^2} + \frac{n_e\beta_e}{1 + \beta_e^2} \right).
\]
The second equality of equation (7) is derived from the first equality using charge neutrality \( n_i = n_e \).

**Hint:** I was able to simplify the algebra by taking, without loss of generality, \( \vec{B} = [0, 0, B] \) and \( \vec{E}' = [E'_\perp, 0, E'_\parallel] \). Then use equation (3) to solve for \( \vec{u}_i = [u_{i1}, u_{i2}, u_{i3}] \), likewise equation (4) for the electrons, and use these to find \( \vec{j} \). Finally cast your answer in the coordinate-independent language of equation (5) by recalling your coordinate choices for \( \parallel \) and \( \perp \).

Follow the hint and write out all the components of the ion equation of motion:
\[
eE'_\perp + (e/c)u_{i2}B = pm_i\gamma_iu_{i1}
\]
\[
-(e/c)u_{i1}B = pm_i\gamma_iu_{i2}
\]
\[
eE'_\parallel = pm_i\gamma_iu_{i3}
\]
The \( u_{i3} \) equation decouples from the others so let’s solve for \( u_{i3} \) first and use it to solve for the field-aligned current (the first term in equation (5)):
\[
u_{i3} = \left( e/\rho m_i\gamma_i \right) E'_\parallel
\]
\[
u_{e3} = -\left( e/\rho m_e\gamma_e \right) E'_\parallel
\]
\[
\vec{j}_3 = e(n_i\nu_{i3} - n_e\nu_{e3})
\]
\[
= e^2 \left( \frac{n_i}{pm_i\gamma_i} + \frac{n_e}{pm_e\gamma_e} \right) E'_\parallel
\]
\[
= \frac{ec}{B} \left( \frac{n_eB}{m_e\gamma_e} + \frac{n_eB}{m_e\gamma_e} \right) E'_\parallel
\]
\[
= \sigma \parallel E'_\parallel
\]
OK 1/3 done. Now go back and solve the two other equations for \( u_{i1} \) and \( u_2 \). First \( u_1 \):
\[
u_{i1} = \frac{eE'_\perp}{m_i\rho_i\gamma_i \left( 1 + \frac{e^2B^2}{c^2(m_i\rho_i\gamma_i)^2} \right)}
\]
\[
n_i\nu_{i1} = \frac{ec}{B} \frac{n_i\beta_i}{1 + \beta_i^2} E'_\perp
\]
from which it follows (including the electrons, too) that \( j_1 = \sigma_P E'_\perp \). That’s 2/3 done. Next \( u_2 \):
\[
u_{i2} = -\frac{eB}{cm_i\rho_i\gamma_i} \frac{eE'_\perp}{m_i\rho_i\gamma_i(1 + \beta_i^2)}
\]
\[
u_{e2} = \frac{eB}{cm_e\rho_e\gamma_e} \frac{-eE'_\perp}{m_e\rho_e(1 + \beta_e^2)}
\]
Notice how $u_{i2}$ and $u_{e2}$ have the same sign, because the charge comes in squared. This recalls how $E \times B$ drift is in the same direction for particles regardless of the sign of their charge. Proceeding:

$$
n_{i}e u_{i2} = -\frac{e c}{B} \frac{n_{i} \beta_{i}^{2}}{1 + \beta_{i}^{2}} E_{\perp}' \quad \text{(22)}
$$

$$
-n_{e}e u_{e2} = +\frac{e c}{B} \frac{n_{e} \beta_{e}^{2}}{1 + \beta_{e}^{2}} E_{\perp}' \quad \text{(23)}
$$

from which it follows that $j_{2} = \sigma_{H} \hat{B} \times \vec{E}_{\perp}'$, after realizing that $3 \times \hat{1} = \hat{2}$. Now we’re 3/3 done.

(b) [10 points] The low-$\beta$ limit: Suppose $\beta_{i} \ll \beta_{e} \ll 1$ (charged particles are not necessarily well-tied to the field because collisions dominate).\(^2\)

First decide which cross-field current is more important. Which is larger, the second or the third term on the RHS of equation (5)? Drop the smaller term.

Taylor expanding we have $\sigma_{H} \propto 1 - \beta_{i}^{2} - 1 + \beta_{e}^{2} \simeq \beta_{e}^{2}$, while $\sigma_{P} \propto \beta_{i}(1 - \beta_{i}^{2}) + \beta_{e}(1 - \beta_{e}^{2}) \simeq \beta_{e}$. So $\sigma_{P} > \sigma_{H}$ and we can drop the $\hat{B} \times \vec{E}'_{\perp}$ term.

Examine the relative magnitudes of the conductivities of the two terms that you have left, working in the low $\beta_{i} \ll \beta_{e} \ll 1$ limit. Simplify accordingly as much as you can, and solve for $\vec{E}'$ in terms of $\vec{j}$.

We have $\sigma_{P} \simeq \sigma_{\parallel} \simeq (e c/B)n_{e} \beta_{e}$. Define $\sigma = (e c/B)n_{e} \beta_{e}$ (compare to the expression in class). Then

$$
\vec{j} = \sigma \vec{E}'
$$

In the lab frame, the velocity of the neutrals is $\vec{u}$.\(^3\) Write down $\vec{E}$, the electric field seen in the lab frame, in terms of $\vec{E}'$, $\vec{u}$, and other given quantities.

This is a Lorentz transform with velocity boost $-\vec{u}$: $\vec{E} = \vec{E}' - \vec{u} \times \vec{B}/c$

Plug your expression for $\vec{E}$ into Maxwell’s induction equation:

$$
\frac{\partial \vec{B}}{\partial t} = -e \nabla \times \vec{E}
$$

Replace $\vec{j}$ with Ampere’s law:

$$
\nabla \times \vec{B} = 4\pi \vec{j}/c
$$

\(^2\)Even if $\beta_{i}, \beta_{e} \ll 1$, it is still possible that flux freezing can hold. It depends on the length scale. On large enough length scales — i.e., for large enough magnetic Reynolds number (see part c) — flux freezing can hold. Collisions only knock charged particles off field lines on small scales. Moreover, collisions are actually necessary for coupling the charged particles to the neutrals — otherwise the neutrals don’t participate in the magnetized flow, which is what happens with ambipolar diffusion (parts d, e, and f).

\(^3\)I beg your forgiveness for changing notation again. This velocity $\vec{u}$ (no subscript) is a lab-frame velocity, of the neutrals only. It should not be confused with $\vec{u}_{i}$ and $\vec{u}_{e}$, which for this problem are ion and electron velocities RELATIVE to the neutrals.
and use the vector identity
\[ \nabla \times (\nabla \times \vec{B}) = \nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B} \] (26)
and the fact that there are no magnetic monopoles (in this universe)
\[ \nabla \cdot \vec{B} = 0 \] (27)
to write down a formula for \( \partial \vec{B}/\partial t \) in terms of \( \vec{u} \) and \( \vec{B} \) and whatever conductivities and constants you need.
\[
\frac{\partial \vec{B}}{\partial t} = \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B} + \nabla \times (\vec{u} \times \vec{B})
\]
This low \( \beta \) limit represents the regime of Ohmic diffusion. Is Ohmic diffusion important in high density or low density environments?

High density for two reasons. First, to drive down \( \beta \propto 1/\rho \). Second, to drive down the fractional ionization \( f_e = n_e/n \) (where \( n \propto \rho \) is the number density of neutrals) since the Ohmic diffusion term scales as \( 1/\sigma \propto n/n_e \propto 1/f_e \). Increasing the overall density tends to decrease \( n_e/n \); this is true not just for thermal ionization (Saha equation predicts fractional ionization to decrease with increasing density) but also for non-thermal ionization processes like photo-ionizations or cosmic-ray ionizations. In all of these cases, the rate at which ions and electrons recombine (reducing the fractional ionization) scales with density to a power greater than that characterizing ionizations (e.g., in photoionization equilibrium, the radiative recombination rate between electrons and ions goes as \( n^2 \), steeper than the photoionization rate which goes as \( n \); and in thermal ionization, the rate of three-body recombinations goes as \( n^3 \), again steeper than the rate of two-body collisional ionizations which goes as \( n^2 \)).

Full credit if just one reason is given.

(c) [2 points] Define a “magnetic Reynolds number” by comparing the order of magnitude of the flux-freezing term (put this in the numerator) to the order of magnitude of the Ohmic diffusion term (put this in the denominator). Show that a sensible definition is:
\[
Re_M \equiv \frac{Lu}{\eta}
\] (28)
where \( \eta \equiv c^2/(4\pi\sigma) \) is the magnetic diffusivity, and \( L \) and \( u \) are characteristic length and velocity scales for the problem. High \( Re_M \) flows behave in the ideal MHD limit where magnetic flux is conserved; for low \( Re_M \) flows, the magnetic field dissipates by diffusing away.

Replace \( \nabla \) with \( 1/L \) to find the desired relation.

(d) [10 points] The high \( \beta \) limit: Now assume \( \beta_e \gg \beta_i \gg 1 \) (particles are well-tied to the field).

As in part (b), first decide which cross-field current is more important. Which is larger, the second or the third term on the RHS of equation (5)? Drop the smaller term.

We have \( \sigma_H \propto 1/\beta_i^2 \) and \( \sigma_P \propto 1/\beta_i \) so \( \sigma_P > \sigma_H \) and we can drop the \( \vec{B} \times \vec{E}_{\perp} \) term.
Now assume the remaining two contributions to the current density are comparable in magnitude.
Based on the relative magnitudes of the relevant conductivities, decide which electric field component
is larger, \( E'_\parallel \) or \( E'_\perp \). Thus show that (it is sufficient to show that the equation is true):

\[
\vec{E}' \simeq -\frac{(\vec{j} \times \vec{B}) \times \vec{B}}{c^2 \rho_i \gamma_i}.
\]  

(29)

where \( \rho_i \) is the mass density of ions. (Hint: only one component of \( \vec{j} \) will survive the cross product above.)

We have \( \sigma_\parallel \gg \sigma_P \) (much easier to slide along the field line than to cross it) and therefore,
assuming \( j_\parallel \sim j_\perp \), we have that \( E'\perp \gg E'_\parallel \). Thus \( \vec{E}' \simeq \vec{E}'_\perp = \vec{j}_\perp / \sigma_P \).

Now consider the desired equation (29). We will plug in \( \vec{j} = \sigma_P \vec{E}'_\perp + \sigma_\parallel \vec{E}'_\parallel \) to show that equation (29) is true. Plugging in, we see that only \( \sigma_P \vec{E}'_\perp \) survives the cross product. Recall we simplified the algebra by taking \( \perp \) to be in the \( \hat{i} \) direction and \( \vec{B} \) to be in the \( \hat{3} \) direction. So we have:

\[
\vec{E}' =? -\frac{(\sigma_P \vec{E}'_\perp \times \vec{B}) \times \vec{B}}{c^2 \rho_i \gamma_i} \\
=? +\sigma_P \vec{E}'_\perp B^2 \hat{2} \times \vec{B} \\
=？ +\sigma_P \vec{E}'_\perp \frac{B^2 \hat{1}}{c^2 \rho_i \gamma_i} \\
\]  

(30)

(31)

(32)

Since \( \beta_e \gg \beta_i \gg 1 \), we have \( \sigma_P \approx (ec/B)n_i/\beta_i = c^2 n_i m_i \rho \gamma / B^2 = c^2 \rho_i \rho \gamma_i / B^2 \). Plug this into the above:

\[
\vec{E}' =? +\frac{(c^2 \rho_i \rho \gamma_i / B^2) \vec{E}'_\perp B^2 \hat{1}}{c^2 \rho_i \rho \gamma_i} \\
\vec{E}' =? \vec{E}'_\perp \hat{1} \\
\]  

(33)

(34)

which is true, since we showed it above when considering the relative magnitudes of \( \sigma_P \) and \( \sigma_\parallel \).

As in (b), Lorentz transform back to the lab frame to find \( \vec{E} \), replace \( \vec{j} \) with \( \vec{B} \) using Ampere’s law, and insert into the induction equation to write down a formula for \( \partial \vec{B} / \partial t \) in terms of \( \vec{\Omega} \) and \( \vec{B} \) (you may compare your answer to equation 27.12 of Shu).

This procedure will yield the same \( \nabla \times (\vec{u} \times \vec{B}) \) field-freezing term as in (b), plus a new non-ideal term that looks like:

\[
-c \nabla \times \vec{E}' = -c \nabla \times \left[ \frac{-(\vec{j} \times \vec{B}) \times \vec{B}}{c^2 \rho_i \rho \gamma_i} \right] \\
= +c \nabla \times \left[ \frac{((\nabla \times \vec{B}) \times \vec{B}) \times \vec{B}}{c^2 \rho_i \rho \gamma_i} \right] \frac{c}{4\pi} \\
= \nabla \times \left[ \frac{B \times (\vec{B} \times (\nabla \times \vec{B}))}{4\pi \rho_i \rho \gamma_i} \right] \\
\]  

(35)

(36)

(37)
which matches the rhs of Shu 27.12.

This is the induction equation with a new non-ideal term: that for ambipolar diffusion. Insofar as it involves two spatial derivatives, this non-ideal term also acts to diffuse away magnetic field, like Ohmic diffusion. But unlike Ohmic diffusion, the diffusivity depends on the quantity being diffused; the diffusivity scales as $B^2$ (see also Shu page 364). So roughly speaking, where $B$ is stronger, the field diffuses away faster.

[Optional (no extra points): Make the connection with diffusion a bit clearer by dispensing with the double cross products in equation (29) and keeping $\vec{v}' \approx \vec{j}_\perp/\sigma P = (c/4\pi)(\nabla \times \vec{B})_\perp/\sigma P$ where we have used Ampere’s law. Thus write down:]

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left[ \frac{c^2}{4\pi \sigma P} (\nabla \times \vec{B})_\perp \right] + \nabla \times (\vec{u} \times \vec{B}) \quad (38)$$

The diffusion-like term (first term on the rhs) is similar to the Ohmic diffusion term (insofar as both have two spatial derivatives), with $\sigma$ replaced by $\sigma P \propto 1/B^2$; the magnetic diffusivity $c^2/(4\pi \sigma P) \propto B^2$ increases with the magnetic field strength. Notice we have not taken $\sigma P$ out of the curl because it depends on $B$. Compare with the ambipolar diffusion term in equation (28) of Wardle (2007).]

(e) [3 points] In ambipolar diffusion, the Hall parameters $\beta_e$ and $\beta_i$ are both $\gg 1$, which means both electrons and ions are well-tied to the field. So when the field diffuses away, it carries both electrons and ions with it—hence the term “ambipolar,” meaning of both polarities. The magnetized charged plasma diffuses away and leaves the neutrals behind. Ambipolar diffusion makes possible, e.g., star formation (enables neutral gas to slip out of field).

We can estimate the relative ion-neutral slip speed—recall we have been calling this $u_i$—as follows. Return to the ion and electron equations of motion (3) and (4) to derive:

$$\frac{\vec{j} \times \vec{B}}{c} = \rho_i \rho_i \gamma_i \vec{u}_i \quad (39)$$

where we have dropped the electron drag term which is small compared to the ion drag term (mostly because $\mu_i \gg \mu_e$, the fact that $\gamma_i > \gamma_i$ notwithstanding; also $u_i \sim u_e$ because both are well-tied to the field, notwithstanding the usual tiny ion-electron slip velocity that generates current).

Multiply equations (3) and (4) by $n_i (= n_e)$ and add to obtain the equation above, dropping the electron drag term as instructed.

Use Ampere’s law again to find that, to order-of-magnitude,

$$u_i \sim \frac{B^2}{4\pi \rho_i \rho_i \gamma_i L} \quad (40)$$
There are astrophysical systems where this drift velocity can be large, e.g., comparable to sound speeds (e.g., Wang & Goodman 2017).

Replace $\vec{j} \times \vec{B}$ with $(c/4\pi)(\nabla \times \vec{B}) \times B$ with $(c/4\pi)B^2/L$ to find the desired order-of-magnitude relation.

(f) [3 points] By analogy to how we defined the dimensionless magnetic Reynolds number $Re_M$ in (c), define another dimensionless number, $Am$, that compares the magnitude of the flux-freezing term to the ambipolar diffusion term in the induction equation that you solved for in part (d). Show that to order-of-magnitude,

$$Am = \frac{u^2 t}{v_A t_{ni}}$$

where the Alfvén speed $v_A = \sqrt{B^2/4\pi\rho}$ (where $\rho$ is the density of neutrals), $t = L/u$ is a characteristic timescale in the flow, and $t_{ni} = 1/(\rho_i\gamma_i)$ is the timescale for a single neutral to collide with an ion in a sea of ions. Thus conclude that if $u \sim v_A$ (true if ram pressures $pu^2$ are comparable to magnetic pressures $B^2$), then $Am \ll 1$ (flux freezing fails and ambipolar diffusion is important) if $t_{ni} \gg t$. This makes sense—if a neutral can’t find an ion to collide and share momentum with before the flow changes, then the neutral gets left behind (and notice how we are now talking about a given neutral in a sea of ions, not a given ion in a sea of neutrals—the latter is controlled by the ion Hall parameter).

Take the magnitude of the flux-freezing term $\nabla \times (\vec{u} \times \vec{B}) \sim uB/L$ and divide by the magnitude of the ambipolar diffusive term $(37) \sim B^3/(L^24\pi\rho_i\rho\gamma_i)$:

$$Am \sim \frac{uL4\pi\rho_i\rho\gamma_i}{B^2}$$

$$\sim \frac{uL4\pi\rho_i\rho\gamma_i}{B^2\rho}$$

$$\sim \frac{u^2 L\rho_i\rho\gamma_i}{u v_A^2 \rho}$$

$$\sim \frac{u^2 L\rho_i\rho\gamma_i}{v_A^2}$$

$$\sim \frac{u^2 t}{v_A t_{ni}}$$

Bottom line: is ambipolar diffusion significant in low density or high density environments?

Low density to make the ion-neutral collision time $t_{ni}$ super-long.

(g) [2 points] There is a third non-ideal regime. What is true about $\beta_e$ and $\beta_i$ here? This is called the Hall diffusion regime.

There is only stone left unturned: when $\beta_e \gg 1$ and $\beta_i \ll 1$.