1 Practice, Practice, Practice

The violin box is a Helmholtz resonator. The box has two f-shaped holes (“f-holes”) carved into the top, of length \( \sim 10 \text{ cm} \) and width \( \sim 7 \text{ mm} \). The thickness of the wood is about 2 mm.

Estimate the fundamental Helmholtz frequency.

We were given the formula for the fundamental frequency of a Helmholtz resonator in class:

\[
\nu_0 = \frac{c_s}{2\pi} \sqrt{\frac{A}{V_0(t + 0.8d)}}
\]

where \( c_s = 350 \text{ m/s} \), \( A = 13 \times 0.7 \text{ cm}^2 \), \( t = 2.7 \text{ mm} \), \( d = 7 \text{ mm} \), and \( V_0 = 1.6 \text{ liters} \). The “0.8d” here adjust for the fact the air around the holes should be affected, too; using the rule-of-thumb from class, I estimate that the air within a distance of 0.4d of the hole should be included, giving a total correction of \( 2 \times 0.4 = 0.8 \). Plugging in, I get that \( \nu_0 = 500 \text{ Hz} \). Actual measurements place the frequency at closer to 300 Hz.

2 It’s Always Greener on the Other Side of the Fence

A demonstration on Cal Day: A 5-gallon transparent drum is upturned over a patch of grass in bright sunshine. Estimate from first principles how long it takes for the grass to significantly reduce the \( \text{CO}_2 \) within the drum. \( \text{CO}_2 \) constitutes about 350 ppmv (parts per million by volume) of the atmosphere.

5 gallons is 0.02 m\(^3\), so I’ll guess that the drum is 30 cm tall with an opening of radius \( R = 15 \text{ cm} \). The mass of \( \text{CO}_2 \) in the drum is

\[
m_{\text{CO}_2} = \rho_{\text{CO}_2} V_{\text{CO}_2} \approx 350 \cdot 10^{-6} \rho_{\text{CO}_2} V.
\]

The density of \( \text{CO}_2 \) is 2 kg/m\(^3\), so

\[
m_{\text{CO}_2} \approx 10^{-8} \text{ g}
\]

We need to figure out how long it takes the grass under the drum to consume that much carbon dioxide. How fast does grass grow? From my lawn care days, I remember that grass grows about \( h = 2 \text{ cm} \) or so in a week. The density of grass is about 0.5 g/cm\(^3\) (it floats), and the blades can
be modeled as cylinders with $r = 1$ mm spaced with maybe $\sigma = 10$ blades per square centimeter. In one minute, therefore, the mass of grass that grows under the drum is

$$\dot{m} = \frac{\rho_g (\pi r^2) h \sigma (\pi R^2)}{7 \cdot 24 \cdot 60} \approx 2 \cdot 10^{-8} \text{g s}^{-1}$$

Now, plants are mostly (90%) water, and maybe half of the remainder is carbon, so the rate at which carbon is removed from the tank is that times $f = 0.1 \cdot 0.5 \approx 10^{-9} \text{g s}^{-1}$ (don’t worry about stoichiometry - this is OOM). The time it takes for the CO2 to get depleted, then, is

$$\frac{m}{\dot{m}} \approx 10 \text{ minutes}$$

Plausible, considering this was a demonstration performed at Cal Day.

### 3 Let’s Do the Twist

*Venus has a mass $M$ and radius $R$ similar to the Earth’s. Its orbit is located 0.7 AU from the Sun, it rotates retrograde with a period of 240 days, and it has neither oceans nor a moon.*

(a) Estimate the timescale, $\omega/\dot{\omega}$, over which tides raised by the Sun on Venus change the Venusian angular rotation rate $\omega$.

From class, we know that

$$\dot{\omega} = \frac{G \rho}{Q} \left( \frac{M_{\text{Sun}}}{M_{\text{Venus}}} \right)^2 \left( \frac{R_{\text{Venus}}}{r} \right)^6$$

Taking $Q \sim 10$, and plugging in the numbers for Venus, I get that

$$\dot{\omega} \sim 5 \cdot 10^{-22} \text{ s}^{-1}$$

The rotational period of Venus is 240 days, so

$$\frac{\omega}{\dot{\omega}} \sim \left[ 10^7 \text{ years} \right]$$

(b) Assume such tides have been acting over the 4.6 Gyr age of the solar system. Estimate the initial rotation period of Venus 4.6 Gyr ago, $2\pi/\omega_{\text{init}}$.

The initial angular frequency is given by

$$\omega_i = \omega - \dot{\omega} \Delta t$$

So, extrapolating back 5 billion years, I find that Venus was originally spinning the other way, with a rotational period of $1$ earth day.

(c) Venus, as with all rocky planets, is thought to have formed by accreting planetesimals of mass $m \ll M$. Assume planetesimals collide with the young Venus inelastically, and that their orbital eccentricities and inclinations prior to impact are large, on the order of unity. Further assume planetesimals to have a single size $r$ and mass $m$.

Estimate $r$ such that the nascent Venus spins at the correct initial rate. Provide both a symbolic and numerical answer.
This is a random walk problem. Each planetesimal brings the growing Venus a certain amount of angular momentum $\Delta|\vec{L}|$. After $N = M_{\text{Venus}}/m$ planetesimals have accreted to form Venus, the nascent planet has a total angular momentum $|\vec{L}| \approx \sqrt{N \times \Delta|\vec{L}|}$. We calculated the initial angular momentum in part b). Thus we have

$$m \approx \frac{(\Delta|\vec{L}|)^2}{(4/25)M_{\oplus}R_{\oplus}^4\omega^2}$$

as the planetesimal mass.

Now what about $\Delta|\vec{L}|$? Estimation of $\Delta|\vec{L}|$ is complicated by several factors. The first is that the planet’s radius is growing with time. This is not too serious because most of the mass of a sphere is located at large radii. The last doubling in mass (during which the last half of planetesimals arrive) involves a fractional growth in radius of $2^{1/3} \approx 1.25$. Venus’s gravity provides another complication, as it will tend to curve the impact path of planetesimals towards the center of Venus, reducing the effective impact parameter $b$. Taking this together with the varying size of Venus, the effective average radial cross section of Venus will be reduced somewhat, probably to somewhere around $R_{\text{eff}} \approx 3/4R_{\oplus}$, the radius at half mass, which we could safely neglect, but we’ll keep anyway since all of these little factors may add up. Another complication is that the planetesimals arrive with random directions onto a planet whose rotation axis is constantly changing. A planetesimal that brings Venus $\Delta\vec{L}$ angular momentum will not, on average, change the magnitude of Venus’s momentum by $|\Delta\vec{L}|$, but rather some fraction thereof. What is $\langle \Delta|\vec{L}| \rangle$ after each impact?

Each planetesimal brings in angular momentum $\Delta L = |\Delta\vec{L}| \approx mvib$, where $b$ is the impact parameter, the effective moment arm, measured relative to some fixed inertial axis passing through the center of Venus. Imagine the still forming planet has transient angular momentum $\vec{L}_0 = (L_x, L_y, L_z)$. Consider an incoming planetesimal with impact parameter $b$. It could be coming from any direction, but since we are free to set axes however we want (as long as we rotate $\vec{L}$ appropriately), let’s say it has angular momentum $\Delta\vec{L} = (0, 0, \Delta L)$ around the center of the planet. After impact, Venus’s angular momentum is $\vec{L} = (L_x, L_y, L_z + \Delta L)$. Now,

$$\Delta|\vec{L}| = |\vec{L}| - |\vec{L}_0| = \sqrt{L_x^2 + L_y^2 + L_z^2} - \sqrt{L_x^2 + L_y^2 + (L_z + \Delta L)^2}.$$ 

If we assume $\Delta L$ is small compared to $L_x$, $L_y$, and $L_z$, we can expand the second square root and keep only the first term to find

$$\Delta|\vec{L}| \approx \frac{L_z \Delta L}{|\vec{L}_0|}.$$ 

Impacts are random, and thus $L_z/|\vec{L}_0|$ and $\Delta L$ should be mostly independent. Thus

$$\langle \Delta|\vec{L}| \rangle \approx \frac{L_z}{|\vec{L}_0|} \langle \Delta L \rangle \approx \langle \Delta L \rangle / \sqrt{3},$$

since, on average, we should expect $L_x$, $L_y$, and $L_z$ to contribute equally to $\vec{L}$, due to the symmetry of the problem. Thus, we see this is a fairly small effect, as $1/\sqrt{3} \approx 3/5$. Nonetheless, these factors are adding up.

Geometric considerations show that the probability that a planetesimal has impact parameter $b$ is $p(b) \approx 2b/R_{\text{eff}}^2$. Thus $\langle b \rangle \approx (2/3)R_{\text{eff}} \approx (1/2)R_{\oplus}$. This means

$$\langle \Delta|\vec{L}| \rangle \approx \frac{1}{\sqrt{3}}mv_i \langle b \rangle \approx \frac{1}{4}mv_i R_{\text{Venus}}.$$
Plug this into the expression for \( m \) above and solve to find

\[
m \approx \frac{64}{25} M_{\text{Venus}} \left( \frac{\omega R_{\text{Venus}}}{v_i} \right)^2 \approx 3 \cdot 10^{24} \text{ g}.
\]

The planetesimals have \( \rho \approx 3 \text{ g/cm}^3 \), which means \( r \approx \boxed{600 \text{ km}} \). That is one big rock; only several hundred of them would be required to form Venus.

4 Talking Smack

_The Spartans fought the Persians at Thermopylae, a mountain pass about 100 meters wide. According to the Greek historian Herodotus, when a Spartan soldier was informed that the Persian arrows would be so numerous as “to block out the Sun,” he replied, “So much the better…then we shall fight our battle in the shade.”_

_Decide whether the Persian threat of arrows darkening the sky is credible._

A good archer can shoot an arrow something on the order of 100m. Allowing for a 50 m buffer zone between the Greeks and the Persians, that means that the Persian archers could cover an area of 50 m \( \times \) 100 m \( = 5 \cdot 10^3 \text{ m}^2 \) without running the risk of hitting their own troops. They are spaced maybe two archers per square meter, meaning that there could have been maybe \( 10^4 \) people firing arrows at the Spartans on that day. Those archers would fire their arrows into a volume of sky of about 20 m \( \times \) 100 m \( \times \) 50 m \( = 10^5 \text{ m}^3 \). How many arrows would their have to be in that volume before it became opaque?

Well, each arrow is maybe 50 cm long by 1 cm wide. Looked at from a glancing angle, each arrow presents a cross-section of maybe \( \sigma = 25 \text{ cm}^2 \). In order for the optical depth of arrows to be one, the number density would then have to be

\[
n = \frac{1}{D \sigma} \approx \frac{1}{20 \text{ m} \times 25 \cdot 10^{-4} \text{ m}} \approx 20 \text{ arrows/m}^3
\]

But in a volume of \( 10^5 \text{ m}^2 \), that implies that we’d need 2 million arrows in the air at a time, fired by only 10,000 archers! Clearly the claim is not plausible.

5 Riddle of Giza

_The Great Pyramid of Giza, built to entomb Egyptian pharaoh Khufu, stands about 150 meters tall. Its base is a square of length 230 m._

_Estimate the number of workers required to build the Great Pyramid._

The volume of the pyramid is \( \frac{1}{3} Ah \) and its center of mass is \( \frac{1}{4} \) of the way up, as you can easily prove by doing an integral. The gravitational potential energy required to assemble the pyramid, then, is

\[
P.E. = \rho V g \frac{h}{4} = \frac{1}{12} \rho g Ah^2
\]
We should also include the energy required to transport all those stones from the quarry to the construction site. The coefficient of rolling friction might be \( \mu \sim 0.1 \) and the distance \( D \) to the quarry might be 30 km. The work done against friction is therefore

\[
W = \rho V g \mu D = \frac{1}{3} \rho g A h D \mu
\]

The total energy to build the pyramid, then, is

\[
E = \frac{1}{12} \rho g A h (h + 4 \mu D)
\]

What about the workers? A human’s base metabolic rate is 100 W, and if we work hard we’re capable of exerting maybe 5 times that for an extended period. The people who built the pyramids, however, were probably weak and malnourished, so let’s say they were only capable of operating at twice the normal BMR. That’s an extra 100 W with which to do work, and factoring in a 25% muscle efficiency tells us that the power available to build the pyramid is

\[
P = N \times 25W.
\]

The pyramid was built during the reign of one guy, so let’s say that it took maybe \( T = 20 \) years to build. The number of workers \( N \) required would then be

\[
N \sim \frac{\frac{1}{12} \rho g A h (h + 4 \mu D)}{25T} \approx 20,000 \text{ workers}
\]

This is in line with estimates made by modern day Egyptologists.

6 Fire Up the Grill

A typical barbeque lighter uses a piezoelectric crystal to create a spark. A pair of electrodes mounted on opposite sides of the crystal connect to wires whose ends are nearly (but not) touching. (The air gap between the wires is just like the gap in an automobile spark plug.) When the crystal is squeezed hard enough, a spark jumps between the wire ends.

Take the crystal to be a cube of length 1 cm, and the gap distance to be 3 mm. Estimate from first principles the amount of force one needs to apply to the crystal to create a spark.

Eugene gave us the rule for piezoelectric materials in class:

\[
\left( \frac{\Delta V}{1V} \right) = 10^9 \left( \frac{\Delta L}{1 \text{ cm}} \right)
\]

We estimated the breakdown voltage for air in the notorious microwave question from PS 3. The argument goes something like this: we’d like to know what electric field is required to accelerate an electron to an energy of \( \sim 1 \text{ eV} \) in a distance of a mean free path. When that condition is met, every time an electron gets knocked off, it will gain enough energy to ionize another atom before it’s expected to suffer a collision. So we need

\[
e Ed \sim 1 \text{ eV}
\]
which gives a breakdown field of $3 \cdot 10^7 \text{ V/m}$ - an order of magnitude too high, it turns out. Using the correct value of the field, we find that

$$\frac{\Delta L}{L} \sim 10^{-5}$$

The force required to produce that strain is

$$F \sim M_Y \epsilon (1 \text{ cm}^2)$$

and using our standard 100 GPa for the modulus we get an answer of [20 N or 5 pounds] of force.

7 Saving Private Ryan

To what minimum depth of water would a World War II frogman (combat swimmer) need to dive to avoid being killed by machine gun bullets fired from the beaches of Normandy?

This is exactly like the cliff diving question from problem set 7, where we found that the stopping distance for a cliff diver upon entering the water is

$$d \sim \frac{2m}{\rho C_D A}$$

A typical calibre bullet has a diameter of 9 mm, a mass of 5 g, and drag coefficient in the ballpark 0.1. Plugging in, I find a distance of a few feet. As many of you pointed out, this problem has been addressed in not one but two Mythbusters episodes. They got the same result for 9 mm rounds, although interestingly with more powerful guns the bullet often shatters upon hitting the water, so more powerful guns actually don’t require as much water for safety.

8 Best Served Cold

A typical computer hard drive consists of a disk (“platter”) of diameter 3.5 inches. Data is stored in concentric tracks (like tree rings) of magnetic patterns: 1’s and 0’s stored as patches of alternating magnetization.

Assume the hard disk has reached the maximum storage density set by the “superparamagnetic limit.” At this limit, the magnetic patches cannot be made any smaller, lest their magnetizations spontaneously flip, corrupting data.

(a) Estimate the maximum storage density in units of bits per square inch.

The superparamagnetic limit happens when the magnetic energy stored in the domain is roughly equal to that domain’s thermal energy, or when

$$\frac{B^2}{8\pi} A d \sim kT$$

Here, B is the field inside the magnet - about 0.1 Tesla - A is the area of a domain, and d is thickness of the magnetized layer, which I’ll take to be a few angstroms. Plugging in and solving
for A, I get that the domains are about $4 \cdot 10^{-11}$ cm$^2$ in size. Since each domain corresponds to one bit, that’s equivalent to an energy density of

$$\frac{1}{4 \cdot 10^{-11} \text{ cm}^2} \times \frac{1 \text{ Gbit}}{1024^2 \text{ Bits}} \times \left(\frac{1 \text{ cm}}{0.4 \text{ inches}}\right)^2 \approx 100 \text{ Gbits/in}^2$$

The all-knowing Wikipedia lists the same figure.

(b) Data are recorded and retrieved through “read/write heads.” The read/write head can be thought of as the needle for a record (LP) player, except that (1) the head does not touch the platter, and (2) data is organized into concentric tracks (not a continuous spiral as on a record). A servo motor moves the head to different radii to sample different tracks. Different azimuths (angles) are sampled by virtue of the platter’s rotation.

Does the servo motor that controls the head’s position in radius need to actively compensate for thermal expansion of the platter throughout the day? Give a quantitative figure of merit that explains why or why not.

The characteristic size of a bit is $\sqrt{A} \approx 60$ nm. The arm that holds the read/write head at the appropriate position must be a few cm long. Since we know (or else Purcell’s cheat sheet tells us) that objects expand by a few parts in 100,000 per degree K, then a mere 1 degree increase in temperature corresponds to an shift of

$$\frac{3 \text{ cm} \times 2 \cdot 10^{-5}}{60\text{ nm}} \sim 10 \text{ bits}.$$  

This ratio of 10 will serve as our figure of merit; since it’s greater than one, it is important to adjust for thermal expansion in your hard drive.