1 Swimming in Syrup

From Gettelfinger and Cussler’s research paper, “Will Humans Swim Faster or Slower in Syrup?”, published in the American Institute of Chemical Engineers Journal in 2004:

“When one of us was training for the 100-m butterfly in the US Olympic Trials, we began to discuss the fluid mechanics of swimming. We noted that swimmers go faster in salt water than in fresh water because they are more buoyant. We argued about how drag could be minimized coming off a turn. Most of all, we wondered whether swimmers would go faster or slower if the viscosity of the fluid was increased [while keeping the density fixed].

“We discussed this with our colleagues, but found no consensus. Most, including some who were experts in fluid mechanics, felt that the swimmers would go more slowly. Some said the swimmers would go faster, because of increased drag on the hands. A few suggested that there would be no change.”

Gettelfinger and Cussler actually performed the experiment by pouring 310 kg of guar gum into a 650 m$^3$ pool. The viscosity of the aqueous guar solution was twice that of water. The density was within 1 part in 10000 of that of water. Sixteen swimmers (10 competitive, 6 recreational) were asked to swim one 25-yard length in a normal pool, two 25-yard lengths in the guar-dosed pool, and finally, after a shower, another 25-yard length in the normal pool. Swimmers rested 3 minutes between each length.

Without looking up the results, predict the outcome of the experiment: faster, slower, or the same in syrup vs. water, and by what fractional amount. Be sure to account for noise in the experiment. E.g., if the answer is faster, estimate the signal-to-noise of the measured fractional difference.

Some of the experts polled by Gettelfinger and Cussler said that the higher viscosity will increase the drag force on the swimmer’s arms, enabling her to throw more water backwards with each stroke and therefore move faster. That is true, but doing so would certainly require more power from the swimmer as well. It makes more sense to assume that the swimmer goes as hard as she can whether she swims in water or guar syrup. In that case, there is a fixed amount of power available, and the speed is set by $P = F_{\text{drag}} \times u$. We need to decide what viscosity has to do with that.

The drag can be thought of as consisting of two components: pressure drag and skin friction. The pressure drag can’t depend on the viscosity, but the skin friction most certainly can, and the extent to which it does in this case depends on the Reynolds number. Let’s estimate the Reynolds...
number for the flow of water over the body of the swimmer, to decide whether the boundary layer is laminar or turbulent.

Remember that the critical Reynolds number for the transition to turbulence is $\sim 2000$ if one evaluates $\text{Re} \text{ using } \delta$ (the thickness of the boundary layer) and $\sim 2000^2$ if one evaluates $\text{Re}$ instead using $x$ (distance along the plate). The two statements are equivalent because $\text{Re}x \sim \text{Re}^2 \delta$ (as Eugene asked us to show in his email).

Imagine a swimmer moving through water at a typical swimming speed of 1 m/s$^1$. At that rate, it would take 25 seconds to swim one of the laps in Gettelfinger and Cussler’s experiment. The Reynolds number is then given by

$$\text{Re} = \frac{ux}{\nu} = \frac{1 \text{ m/s} \times 2 \text{ m}}{10^{-6} \text{ m}^2/\text{s}} \approx 2 \times 10^6.$$ 

The flow is therefore turbulent and we expect pressure drag to dominate. However, we’re actually fairly close to the laminar/turbulent border here — at a length scale of 0.2 m, the Reynolds number is $2 \times 10^5$ and we’re back in the laminar regime. Let us therefore say that there is a laminar boundary near the head of the swimmer, with becomes turbulent after about 0.2 m.$^2$

There should therefore be three terms contributing to the drag: a (dominant) pressure term, a viscous friction term due to the laminar boundary layer near the swimmer’s head, and a turbulent skin friction from the flow of water over the rest of the body. The pressure drag is given by $\frac{1}{2} \rho A C_D u^2$. Humans haven’t evolved to be especially hydrodynamic, so let’s say the drag coefficient is $\sim 1$. The cross-sectional area of a person is maybe like 0.1 m$^2$, so the pressure drag evaluates to

$$0.5 \times 1000 \text{ kg/m}^3 \times 0.1 \text{ m}^2 \times 1 \times (1 \text{ m/s})^2 \approx 50 \text{ Newtons.}$$

The viscous drag, on the other hand, is given by $\rho \nu \frac{\delta}{2} A$, where $\delta$ is the skin depth and $A$ is the surface area. Using $\delta = \sqrt{\frac{\nu x}{u}}$ I get that

$$F_{\text{skin}} = \rho \nu \frac{1}{2} u^3 x \frac{1}{2} b \approx 0.5 \text{ Newtons}$$

where I’ve used $x = 0.2 \text{ m}$ and $b \sim 1 \text{ m}$ to account for the swimmer’s shoulders and wingspan.

Finally, there will be a turbulent skin friction $F_{\text{fudge}} \rho u^2 A$, where $F_{\text{fudge}}$ is $10^{-3}$. For a surface area of 1 m$^2$, this gives an additional drag force of

$$10^{-3} \times 10^3 \times 1^2 \times 1 = 1 \text{ Newton.}$$

So, the viscosity-dependent portion of the drag is less than 1% of the total, at a speed of 1 m/s. For faster swimmers, the skin drag is even less important, since it scales more slowly with $u$ than does the pressure drag. Clearly, viscosity must be a small effect.

Now let’s think about how changing the viscosity might affect a swimmer’s speed. The power required to maintain a speed of 1 m/s in regular water is $F_{\text{drag}} \cdot u = 50 \text{ W} + 0.5 \text{ W} + 1 \text{ W} = 51.5 \text{ W}$. What would happen when our swimmer switches to the gummed-up pool? The density difference is tiny, so let’s ignore that. Let’s also — as discussed above — assume that the power a person is capable of expending remains the same. The dimensions of our swimmer certainly haven’t changed,

$^1$This a *recreational* swimmer we’re talking about here - according to Wikipedia, the fastest 50 m time ever recorded was by Amaury Leveaux of France at 20.48 seconds - an average speed of 2.4 m/s

$^2$This turns out to be about right, according to the actual results of the paper.
so the only relevant variable here is the viscosity. If we double that, we don’t change the pressure
term above, but we double the viscous term, because we pick up a $\sqrt{2}$ from $\nu^2$ and also because
the turbulent transition now happens at 0.4 m. The third term should get a little bit smaller as
well since there would be less wetted area, but that’s a small effect so let’s ignore that. The speed
is then given by the solution to

$$50u^3 W + u^\frac{5}{2} W + 1 W = 51.5 W$$

or $u = 0.997 \text{ m/s}$, a change of a few parts per thousand.

Is that a significant difference, or not? The second part of the question was to compute the
“signal to noise ratio of the measured fractional difference” - i.e. to compare this one part in a
thousand difference to the variation you would expect from lap to lap anyway. Eugene was kind
enough to provide us with some experimental data here. He swam four 25 m laps of Hearst Pool
and measured the following times: 33 s, 33 s, 34 s, 34 s. So it seems like you would expect about
a $1/33 = 3\%$ variation in lap time even without messing with things like viscosity, which you
probably could have guessed anyway. The signal-to-noise ratio is then $0.003/0.03 \approx 0.1$, so we can
“predict” that Gettellinger and Cussler shouldn’t detect any difference between the two pools in
their experiment. According to the paper, that is indeed what they found. I find it surprising
that such a simple analysis can get the right answer, considering all the complicated variables that
you might think should come into play - the size and shape of the swimmer, how high in the water
they are, the exact flow pattern around the (certainly unsteady) motion of the arms, etc.

It’s interesting to think about whether the same holds true for professional swimming, which
is all about tiny differences in performance. Michael Phelps won the 100 m butterfly in Beijing
by only one hundredth of a second, which in a 50 second race is a fractional difference of 0.0002,
or a signal-to-noise ratio of about 7. Apparently viscosity does matter at the very elite level. For
recreational swimmers, however, you are unlikely to notice a difference.

2 Your Mileage May Vary

*Construct a plot of mileage (miles travelled per gallon of gas consumed) vs. speed for a
typical 4-door sedan. Label your plot quantitatively (numbers and slopes).*

*Use order-of-magnitude physics and/or any experimental data that does not include the
actual answer (i.e., do not merely report what your odometer reads after you’ve used up
a tank of gas). You may, e.g., play with toy cars.*

Gas is burned to fight air drag and ground friction. At very low speeds, some gas may be burned
to just keep the engine running while the wheels aren’t turning. Consider air drag first, which we
expect to dominate at high speeds.

Most of that drag is pressure drag, $D = (1/2)C_D p U^2 A$, and the corresponding parasitic power
is $P = D U$. Engines are not perfectly efficient in converting chemical energy into mechanical
energy, so the true power requirement should be divided by the engine efficiency, $P = D U/\epsilon$,
where modern engineering has produced $\epsilon \sim 0.2$ perhaps.$^4$ We assume this efficiency is constant

$^3$available here: http://www3.interscience.wiley.com/journal/109667863/abstract

value of 20% is actually pretty common
at all speeds but the lowest, which is a reasonable assumption for a good transmission, such as a CVT or that found in the Toyota Prius. This assumption isn’t as great for automatic or manual transmission vehicles, but it’s not too bad either.

The energy content of gasoline is like fat, \( q \sim 9 \text{ Cal g}^{-1} = 3.75 \times 10^{11} \text{ erg g}^{-1} \). After driving for time \( t \), the mileage is, by definition,

\[
\frac{Ut}{Pt/q} = \frac{Uq}{P} \approx 30 \text{ mpg} \left( \frac{60 \text{ mph}}{U} \right)^2
\]

where we have used \( A \sim 4 \text{ m}^2 \), \( \rho = 10^{-3} \text{ g cm}^{-3} \), converted 3000 g = 1 gallon of gas, and used \( C_D \sim 0.3 \), based on the handout in class. This is our answer considering only air drag. So far we’re on the right track since at 60 mph we expect air drag to dominate, and 30 mpg is a common highway mileage for sedans.

Do we have to worry about a laminar/turbulent transition changing \( C_D \) at some velocity? I suspect that turbulent/laminar flow does not have a substantial effect on the pressure drag of a car, since we don’t see any attempt to induce turbulence in streamlined vehicles (like the Honda Insight or GM EV1) at low velocities, and it’s unavoidable at medium to high velocities. \( Re_x = UX/\nu \sim U \times 2 \cdot 10^5 \) (mks) for a 4 m long car. Thus at greater than 5 m/s ~10 mph, the flow is turbulent regardless. At less than 10 mph, as we’ll see, pressure drag is a small effect anyway.

![Figure 1: Gas mileage vs. speed for an average sedan.](image)

At low speeds, rolling friction dominates. The frictional force is \( \mu_{\text{roll}} mg \), where \( \mu_{\text{roll}} \) is the coefficient of rolling friction, and \( m \sim 1 \text{ ton} \sim 10^6 \text{ g} \). We can estimate \( \mu_{\text{roll}} \) in a variety of ways. We can play with a model toy car on a ramp. If the ramp is inclined more than a degree or so, the toy car starts to roll. So \( \mu_{\text{roll}} \sim 1^5 \sim 1/60 \sim 0.02 \). Driving my car, I find that on a smooth flat surface, it takes a little over 100 m to stop from 5 mph. This implies \( \mu_{\text{roll}} \sim 0.02 \), including friction

---

\[5\text{This figure checks out with data on automobile drag on Wikipedia.}\]
on the axles. Thus, the frictional force is \( F_{fr} \sim 2 \times 10^7 \text{ dyne} = 200 \text{ N}, \) independent of velocity. This is in accord with my experience that I can push a car on a level smooth surface. Comparing the frictional force with \( F_D \sim 4.5 \times 10^7 (U/60 \text{ mph})^2 \) dyne, we find that at \( U = 60 \text{ mph}, \) ground friction adds another \( \sim 50\% \) to the total power requirement. Furthermore, \( F_{fr} > F_D \) for \( U < 40 \text{ mph}. \)

The total resistance is the sum of ground resistance plus air drag. At \( U = 40 \text{ mph}, \) when there are equal contributions from air and ground friction, the mileage is about 36 mpg. For \( U > 40 \text{ mph}, \) the mileage gets progressively worse, asymptotically behaving as \( 1/U^2 \) at high speeds, due to air resistance. For \( U < 40 \text{ mph,} \) the mileage saturates to a value that is insensitive to \( U \) of about 70 mpg, due almost exclusively to ground friction. Finally, at very low speeds, mileage drops to zero as the engine spins without turning the wheels. Toyota’s Prius cleverly avoids this low speed pitfall by running on battery only (with engine off) when it can. Further, with regenerative breaking, the Prius does indeed obtain near 70 mpg in stop-and-go traffic. Cars without regenerative breaking rarely see such stellar mileage at low speeds because usually low speed comes with lots of breaking and accelerating, which eats up a lot of power. Gas mileage is plotted in Figure 1.

2.1 Update: May 2009

Kevin O’Brien pointed out that real cars actually get the best milage at speeds closer to 40 mph, as the following figure (courtesy of the Federal Highway Administration) shows.

He also correctly identified the problem with the above solution: we left out the power it takes just for the car to idle. So really, we should add in a constant term \( P_{idle} \) to the power and recompute the milage. If I take \( P_{idle} \) to be \( \sim 30 \text{ horsepower}, \) I can reproduce the real curve fairly accurately:
The typical American car has a horsepower of around 120, so what this says is that the power consumption just to keep the engine running is a significant fraction of the total power available to the car. Apparently, just running the alternator and water pump as well as any accessories you might have on really costs you. In retrospect, maybe this isn’t too surprising, since we know that cars idle at a substantial fraction of their maximum rotations per minute. Note also that at 30 HP $\sim 22.5 \text{ kW}$ the idle power requirement is the biggest determinant of the total fuel efficiency until you get close to highway speeds.

2.2 Summary

If I put together all 3 contributions to the power, air friction, ground friction, and idle engine power, and I write the velocity in terms of a non-dimensional $u = \text{velocity} / 60 \text{ mph}$, and pull out all the other constants, I get the following equation:

$$\text{Mileage} = 70\text{mpg} \times \frac{1}{1 + u^2 + \frac{1}{2u}}$$

(1)

The constant term of 1 in the denominator comes from the constant ground friction.

The $u^2$ dependence came from the air drag.

I note that factor of 1/2 in the idle power term $\frac{1}{2u}$ was attained by cheating: Wolfram Alpha told me that the maximum of the nondimensional part would be at a velocity close to 40, the best velocity as discussed above, if I set that factor close to 1/2. This factor of 1/2 corresponds to about $\approx 10-30$ horsepower if I undo the non-dimensionalization.

We wanted you to consider air drag, and additionally note that the behavior of mileage is not monotonically decreasing. That is, you could get full credit either by considering ground friction or idle power.

By putting the equation in this form, it’s easy to note that 70 mpg is an approximate upper limit (without doing any calculus to find the actual maximum mpg at a speed of $\approx 40 \text{ mph}$).
3 Amtrak

Consider a passenger railway in North America travelling at top speed. How many passenger cars could be added to the train before skin friction drag becomes more important than pressure drag?

It’s a pretty sure bet that the boundary layer is turbulent, but just to be sure we’ll estimate $Re_x \sim U X/\nu$, where $U \sim 100$ mph $\sim 4.4 \times 10^3$ cm s$^{-1}$ and $\nu \sim 0.2$ cm$^2$ s$^{-1}$ for air. For the boundary layer to be turbulent, for a surface that’s not particularly smooth, $Re_x > 10^5$, so $X$ must be at least as long as $\sim 10$ cm for the boundary layer to be turbulent. It’s definitely turbulent! (If instead you evaluated $Re_\delta$, the critical number to compare against would be $\sim 2000$.)

From class, the stress exerted by a turbulent boundary layer on a plate is $\sigma \sim (1/2)C_D \rho U^2$ where $C_D \sim 0.002$. From the Course Reader page 87 (Eugene discussed this in his email), we see that this value for the drag coefficient is really appropriate for very smooth surfaces. Since the train is broken up into separate cars and since the surface is not particularly polished (there are rivets and windows), the value for $C_D$ should be higher. The plot by White gives an uppermost value of $C_D \sim 0.012$ for a surface roughness of 1/300; this exceeds our class estimate by a factor of 6; the effective surface roughness of the train is probably bigger than 1/300, so we’ll round up the enhancement factor of $C_D$ to 10. That means $C_D \sim 0.02$ and the stress is $\sigma \sim (1/2)\rho U^2 \times 0.02$.

Call the frontal area of the train $A$. We estimate that each side of train gives an area of $\sim 3A^6$. Multiplying by the four sides of the train car (including the bottom near the tracks) gives a total side area of $\sim 12A$. The skin friction drag for $N$ cars is $N \times 12A \times (1/2)0.02\rho U^2$. The pressure drag is $A \times (1/2)\rho U^2 \times 1$, where the 1 is our estimate for $C_D$ due only to pressure drag for North American trains that are not especially streamlined. Set the drags equal and solve for $N \sim 4$.

This seems of the right order of magnitude. From the Course Reader page 92, the effective total $C_D$ of a North-American-boxy-looking 6-car train is 1.8, while the effective total $C_D$ for a tractor-trailer truck is 0.96 (without the fairing and gap seal). A tractor-trailer truck is not as long as a train, so we can assume that pressure drag dominates for the truck. All of this data is consistent with our above solution which states for a 4-car train, pressure drag contributes $\sim 1$ to $C_D$, while skin friction drag contributes another $\sim 1$.

\[\text{If this estimate makes you squirm, it is just a statement that the car is three times as long as it is wide. Our final answer of 4 cars is equivalent to saying the threshold length of the train is 12 times the width of the train.}\]