1 Walk the Plank

“Yo ho, yo ho, the frisky Plank!
You walks along it so,
Till it goes down, and you goes down
To Davy Jones below,”
says Captain Hook to the captive children as he forces them to walk the pirate’s plank.

Estimate the minimum thickness that the plank must have not to snap under a child’s weight.

Take the plank to have length $l$, vertical thickness $a$, and horizontal thickness $b$. We’ll neglect the mass of the plank (“massless plank”), though it’s not hard to include it in the calculation. We need to estimate $a$. The strain in the plank is of order $\epsilon \sim \Delta l/l \sim a \Delta y/l^2$ from lecture, where $\Delta y$ is the vertical deflection. Also from lecture,

$$F \sim \mu \frac{a^3 b}{4 l^3} \Delta y,$$

where $F = mg$ is the applied force due to the child’s weight. Plugging it all in,

$$\epsilon \sim \frac{4lmg}{\mu a^2 b}.$$

The plank snaps when $\epsilon = \epsilon_F \sim 10^{-3}$ (a typical failure strain for ordinary materials), so the minimum $a$ is

$$a \sim S \sqrt{\frac{4lmg}{\mu \epsilon_F b}},$$

where we have thrown in a safety factor $S \sim 3^1$. For wood, the elastic modulus is $\mu \sim 10^{11}$ erg cm$^{-3}$. For $m$, we take the child’s weight of 30 kg. We assume $l \sim 6$ feet = 200 cm and $b \sim 30$ cm (the span of two human feet). Then $a \sim 3$ cm, which seems about right, from my experience on pirate ships.

\[1^1\text{Plank’s constant}\]
2 Fastball

Estimate using order-of-magnitude physics the speed of a fastball pitch. The elastic modulus of muscle ranges between $10^6$ and $10^7$ dyne/cm$^2$.

Hint: Not all of the work the pitcher does goes into the ball.

There are multiple ways to approach this problem. I have chosen, based on personal bias, to view it in terms of torques. The shoulder muscle torques the arm which holds the ball, thus accelerating the arm. The velocity of the fastball is the velocity of the end of the arm (also called hand, colloquially) at the time of release. The velocity obeys $v = l\omega$ where $l$ is the length of the pitcher’s arm. We also know that the arc that the pitcher throws through measures an order unity angle 1 (in rads), $1 = \omega t$ and that $\omega = at$. Putting it all together, $1 = \frac{1}{2}at^2 = \frac{1}{2}\omega^2/\alpha$ so $\omega = \sqrt{2\alpha}$. To get $\alpha$, we know Torque $\tau = I\alpha$. The moment of inertia for the arm (a rod pivoting about one end) is something like $I = \frac{1}{3}ml^2$ where $m$ is the mass of the arm.

Now for the Torque. The shoulder muscles (where it aches after a lot of throwing) exert a torque on the arm. They have a tension $T$, and this tension is applied on the arm bone at a distance approximately $r$, the radius of the arm at the shoulder, for a torque $\tau = Tr$. We know from class that $T = EA\sigma$ for elastic modulus $E$, cross sectional area $A$, and strain $\sigma$. The mass of shoulder muscle has an area that is probably close to $A \sim \pi r^2$. Recalling that Eugene mentioned that muscles are flexible, I envision the extent of distortion that muscles undergo during flexing. Muscle bulges can sometimes nearly double in size for athletes or for that one friend that everybody has, so I’ll take the strain to be order unity. Just to check, since this is contrary to usual material strains we work with, according to Wikipedia$^2$, the force a muscle can exert per square centimeter is about 30 Newtons, which is equivalent to the given elastic modulus of $10^7$ dyne / cm$^2$ with a strain of 0.3, which is pretty close to 1. Thus, the torque is $\tau = E\pi r^2 \sigma r \sim E\pi r^3$

Putting everything together,

$$\omega = \sqrt{2\alpha} = \sqrt{\frac{2\tau}{I}} \sim \sqrt{\frac{2E\pi r^3}{\frac{1}{3}ml^2}}$$

for $E = 10^7$ dyne/cm$^2$, $r = 10$ cm, $l = 1$ m, and $m = \rho \ast \pi \ast r^2l$ for a water-filled cylindrical arm. This gives a frequency of 17 Hertz. $^3$

$v = \omega l$ so the velocity is $[17 \text{ m/s}]$ or $[38 \text{ miles per hour}]$. This is probably what a single human arm is capable of. It is a factor of $\sim 3$ slower than a 100 mph fastball because of at least two other contributions: the effect of curling and uncurling the arm during the throw, and the fact that the torso also contributes significantly (hence the dramatic windup before any pitcher’s throw) — just like how horses use their entire body when they gallop, not just their leg muscles. The movement of the torso means that the arm itself has some base velocity which must also be added. Multiplying the $E$ by a factor of 10 to account for the extra torso muscle in our equation above for $\omega$ would enable us to recover the missing factor of 3.

$^2$http://en.wikipedia.org/wiki/Muscle#Physiological_strength
$^3$This compares well with a frequency found in http://www.popularmechanics.com/outdoors/sports/physics/how-the-105-mph-fastball-tests-the-limits-of-the-human-body of 24 Hz.
3 Earth Song

There is a giant tuning fork in the traffic median between Shattuck Avenue and Center Street. Estimate its natural frequency using order-of-magnitude structural physics.

The tines of the fork behave like struts. Let’s once again make use of our formula for the restoring force:

$$F \sim \frac{\mu a^3 b}{4 l^3} \Delta y.$$  

Using Newton’s second law, we have

$$m \left( \ddot{\Delta y} \right) \sim \frac{\mu a^3 b}{4 l^3} \Delta y$$  

and we can read off that

$$\omega^2 \sim \frac{1}{m} \frac{\mu a^3 b}{4 l^3} \sim \frac{1}{\pi a^2 l \rho} \frac{\mu a^3 b}{4 l^3} \sim \frac{ab \mu}{4 \pi l^4 \rho}.$$  

Now we just have to guess the properties of the fork. I walk past this thing every day, and I’d say the tines are about 10 m high, with $a \sim b \sim 20$ cm. The density of a typical metal is $\sim 10$ g cm$^{-3}$, and we can use our standard 0.2 Mbar for the Young’s modulus. Plugging everything in, I get that $[\omega \sim 1 \text{ Hz}]$. In fact, if you read the inscription on the sculpture, it says that the fork was designed to oscillate with a frequency of 0.6320 Hertz, which the artist Po Shu Wang says is “tuned to the oscillating frequency of the Earth.” At first I figured he was just being metaphorical, but actually it turns out that after a big seismic event, the Earth is left with a residual “ringing” in a number of different modes. The fork is tuned to be 12 octaves higher than the lowest of these frequencies.