1 The Tuning Fork

A standard tuning fork consists of two tines each 8 cm long, 4 mm in diameter, and separated by 1.25 cm. It emits a nearly pure sine wave at 440 Hz.\(^1\)

A tuning fork is struck hard, then allowed to radiate sound while suspended in free space (that is, it is not coupled to any sounding board). Does the kinetic energy in the tuning fork decay principally due to (a) acoustic radiation, (b) aerodynamic friction, or (c) inelasticity in the metal? Supply OOM arguments.

The kinetic energy of 2 oscillating tines is \(2 \times \frac{1}{2}mv^2\). The mass \(m\) of a single aluminum tine is about \(5 \text{ g cm}^{-3} \times 8 \text{ cm} \times \pi \times (0.2 \text{ cm})^2 \approx 5 \text{ g}\). For a hard strike, I see that a single tine quivers by about 1 mm, so I’ll take \(\xi = 1 \text{ mm} = 0.1 \text{ cm}\) for the amplitude of the oscillation. Over one cycle lasting \(1/f\) in time, where the frequency \(f\) is 440 Hz, each tine sweeps across a distance \(4 \times \xi\). The average velocity is therefore \(0.4 \text{ cm} \times 440 \text{ Hz} \approx 176 \text{ cm/s}\). Thus, the time-averaged kinetic energy of the tuning fork (both tines) is \(5 \text{ g} \times (176 \text{ cm/s})^2/3 \approx 5 \times 10^4 \text{ erg}\), where we have divided by 3 because the effective mass taking part in the full oscillation is only the top 1/3, as we consider the bottom 2/3 firmly rooted to the base of the fork. A spring has equal kinetic and elastic potential energies averaged over time, so the total energy of the fork is \(E \approx 1 \times 10^5 \text{ erg}\).

Our plan will be to estimate the ring-down times due to (a), (b), and (c) and then see how they compare to the actual ring-down time, which we estimate by experiment to be 10 s.

(a) Acoustic Radiation  The ring-down time will be the energy of the tine divided by its acoustic power. Each tine radiates as a dipole (no mass flux, only momentum flux). Two tines oscillating in anti-phase radiate as a quadrupole.

In class we calculated the acoustic power of a pulsating monopolar sphere. Let’s follow that derivation now and calculate the power from a monopolar, pulsing (expanding-and-contracting) cylinder of radius \(r\) and length \(l\). Imagine that the cylinder gets alternately fatter and thinner in 1 direction with amplitude \(\xi\) and frequency \(f\). Afterwards we’ll multiply by two dilution factors: one to bring the power down to a dipole, and another to reduce it to that of a quadrupole.

Examine the energy flux at the near-zone boundary of air surrounding the tine. Inside the near-zone, pressure waves travelling at the speed of sound smooth out density disturbances caused by the breathing tine, and the entire volume of near-zone air moves in phase with the tine. As

\(^1\)The standard frequency for the note A was set at 440 Hz in 1953. In Mozart’s day, the standard A was higher, more like 444 Hz.
estimated in class, the radius of the near-zone is approximately the distance a sound wave can
travel in $\sim 1/(2f)$, or $r_s/(2f) \approx 40$ cm. This is 5 times larger than the largest dimension of the
tine (its length), so it will be OK to think of the near-zone volume as a rough sphere (rather
than a cylindrical volume). Sound waves are launched at the boundary of the near-zone into the
far-field radiation zone. Upon launch, these sound waves contain kinetic energy density of order
$\frac{1}{2}\rho_0 u^2$, where $\rho_0$ is the unperturbed air density and $u$ is the perturbation bulk velocity of air at
the near-zone boundary. The kinetic energy flux is then $F_K = \frac{1}{2}\rho_0 u^2 c_s$, since the wave propagates at
the sound speed $c_s$.

From class,

$$\frac{u}{c_s} \sim \frac{\Delta V}{V}$$

where $V \sim 4(c_s/2f)^3$ is the near-zone volume and $\Delta V \sim \pi r\xi l$ (the expansion-contraction is only
in one direction, hence $\pi$ instead of $2\pi$). Thus,

$$u \sim r\xi l \left(\frac{2f}{c_s}\right)^3 c_s$$

and

$$F_K \sim 32\rho_0 f^6(r\xi l)^2 c_s^{-3}.$$

The kinetic power carried by the sound wave is $F_K$ times the surface area of the near-zone boundary,
$4\pi(c_s/2f)^2$. The wave (read: spring) also carries an equal amount of extra thermal energy in
adiabatically compressed regions, so multiply by 2 to obtain the total power. The total monopolar
power is then

$$P_{\text{mono}} \approx 64\pi\rho_0 f^4(r\xi l)^2 c_s^{-1}.$$

A single tine on a tuning fork does not pulse, though; it sways back and forth, compressing the
air on one side while rarefying it on the other. Model a single swaying tine of dimensions $(a,b,l)$ as
two out-of-phase, pulsating tines each of dimension $(a/2,b,l)$, separated by length $a/2$ (measured
between their centers, so they are just touching). The effective separation characterizing these two
pulsating tines is $d = a/2$, which for our cylindrical tine corresponds to $d = r$. From class, we know
the power is reduced by $[d/(c_s/2f)]^2$. Thus,

$$P_{\text{dipole}} \sim P_{\text{mono}} \left(\frac{2rf}{c_s}\right)^2.$$

Of course, there are two swaying tines, separated by distance $x = 1.25$ cm, so we further reduce
the dipole power by another factor of $[x/(c_s/2v)]^2$:

$$P_{\text{quad}} \sim P_{\text{dipole}} \left(\frac{2xf}{c_s}\right)^2.$$

Putting it all together,

$$P_{\text{quad}} \sim 1024\pi(\xi xl)^2 r^4 f^8 \rho_0,$$

which is a pretty crazy formula.

For the tuning fork, $f = 440$ Hz, $r = 0.2$ cm, $l = 8$ cm, $x = 1.25$ cm, and $\xi \approx 0.1$ cm. For air at
STP, $\rho_0 = 10^{-3}$ g cm$^{-3}$ and $c_s = 3.4 \times 10^4$ cm/s. Thus, $P_{\text{quad}} \approx 2 \times 10^{-4}$ erg/s, and the ring-down
time due to (a) acoustic radiation is $E/P_{\text{quad}} \approx 5 \times 10^8$ s, or more than 10 years! Clearly acoustic
radiation is not responsible for damping the fork.
(b) Aerodynamic Drag  
With aerodynamic drag, the power dissipation is $F_{\text{drag}}v$ per tine, where $v \sim 4\xi f \sim 176 \text{cm s}^{-1}$ is the average velocity of the tine and $F_{\text{drag}} \sim (1/2)C_D\rho_0v^2A$, where the effective area $A \sim (l/3) \times (2r)$, where the factor of 3 accounts for only $1/3$ participation of the tine (see above). To decide $C_D$, evaluate the Reynolds number $Re \sim rv/\nu \sim 100$, which tells us that pressure drag will dominate skin drag, and we can take $C_D \sim 1$ for a blunt tine. Then

$$P_{\text{drag}} \sim F_{\text{drag}}v \sim \frac{1}{3}\rho_0v^3 lr \sim 3 \times 10^3 \text{erg s}^{-1},$$

which makes for a ring-down time of $(E/2)/P_{\text{drag}} \sim 17 \text{s}$ (we took $E/2$ since $E$ is for both tines). This does explain the ring-down time to an OOM.

(c) Inelasticity  
We didn’t really discuss inelasticity in class, but those of you who know about “quality factors $Q$” of oscillators might know that typical $Q$’s are $10^2$ for ordinary materials. For a carefully engineered instrument like the tuning fork, perhaps $Q \sim 10^3$. That means the energy decays in $Q \times (1/f) \sim 2 \text{s}$. This is fast enough to explain the observed ring-down time of $10 \text{s}$.

Our final answer is that inelasticity and aerodynamic drag seem to play comparable roles in damping out the oscillation. Acoustic radiation is not important at all in the energy budget.

2  For Whom the Bell Tolls

*Leslie observed that a bell sounds softer when immersed in air mixed with hydrogen.*

Estimate the factor by which the bell’s acoustic power is reduced when immersed in hydrogen at STP versus air at STP (all other factors remaining the same).

The key is to realize that the near-zone volume is much larger when the acoustic medium is hydrogen, because the sound speed in hydrogen is $c_s \propto \sqrt{\mu_{\text{air}}/\mu_{\text{H}_2}} \sim 4$ times faster than in air due to the difference in mean molecular weight. This much larger near-zone volume dilutes the perturbation.

If we idealize the complicated shell of the bell as a flat metal sheet and imagine striking such a sheet, then the lowest order mode of the sheet vibrates as a dipole (swaying back and forth). We’ll assume the bell radiates as a dipole.

We know, either from class or from the problem above, that the acoustic power radiated by a monopole scales as $P_{\text{mono}} \propto \rho_0/c_s$. Now $\rho_0 \sim P_0/c_s^2$, so $P_{\text{mono}} \propto 1/c_s^3$ (since $P_0$ is fixed at STP). Also, $P_{\text{dipole}} \propto P_{\text{mono}}/c_s^2$. All together, the bell’s acoustic power scales as $P_{\text{dipole}} \propto 1/c_s^5$, and is reduced in pure hydrogen by a factor of $\left(\mu_{\text{air}}/\mu_{\text{H}_2}\right)^{5/2} \sim 700$, or 30 dB.

3  The Burning Glass of the Siege of Syracuse, or “Keck Death Ray”

*The Roman Fleet lay siege to the Greek port city of Syracuse in 214 BC. Legend has it that Archimedes devised a mirror that focused the Sun’s rays to set fire to the Roman ships, “turning them to ashes at the distance of a bow-shot”. Most people discount this story as myth, though it is probably fair to say that there is no definitive answer either way.*

3
(a) We know from Diocles’s treatise “Burning Mirrors” that Greeks in the late second century BC understood that parallel rays are focused to a point by a parabolic mirror. Suppose Archimedes wanted to use a single parabolic mirror. Estimate the diameter of the required mirror.

Those of us who watch “Mythbusters,” know the answer here - it is technically possible to set a boat on fire by reflecting the sun’s rays, but very hard to make this work as a practical weapon, especially with the technology the Greeks had available. Like with the laser pointer problem, the sticking point here is that the light must be directed at a small spot on the ship’s hull in order for the rays to be intense enough to start a fire in reasonable amount of time, which I’ll take to be $\tau \sim 10$ s. Hence, computing the spot size the Siracusans could have obtained is crucial.

Let’s start our analysis by assuming a perfectly parabolic mirror. Parabolas have the interesting property that they focus all incoming light down to a single point - provided, of course, that the rays are coming in strictly parallel to the mirror axis. In reality that isn’t the case, but in our first attempt let’s pretend like that isn’t a problem. Since the mirror could in principle focus all the solar flux down to a tiny point, what is it that sets the spot size for the focused rays?

Well, it’s important to keep in mind that the boat is not perfectly still. Even if it isn’t moving around the surface (and I don’t think the enemy is particularly likely to sit still while we try to set them on fire), there will still be some motion due to waves, which from experience will move the boat up and down vertically over a distance of about 1 m during our exposure time. That means that the spot will spread out over a radius of

$$R_{\text{spot}} \sim 0.5 \text{ m}$$

The mirror, on the other hand, has a light gathering power of

$$F_{\text{sun}} \pi R_{\text{mirror}}^2$$

so the intensity of light on the hull of the boat is

$$F_{\text{sun}} \left( \frac{R_{\text{mirror}}}{R_{\text{spot}}} \right)^2$$

and the energy absorbed over the entire exposure is

$$F_{\text{sun}} \pi R_{\text{mirror}}^2 \tau$$

Now, in order to start a fire, we’ll need to heat the wood past it’s burning point, which as many of you pointed out we know from Ray Bradbury to be 451°F $\sim 500$K. The total amount of wood we’ll heat up is equal the area of the spot times some characteristic depth over which heat is conducted through the material in our exposure time $\tau$. We know the thermal diffusion coefficient of wood from lecture ($\kappa \sim 10^{-2}$ cm$^2$ s$^{-1}$), so that depth is

$$d \sim \sqrt{\kappa \tau} \sim 3 \text{ mm}$$

Hence the total mass of wood we have to heat is

$$\rho \pi R_{\text{spot}}^2 d$$
which will take an energy of \[ \rho \pi R_{\text{spot}}^2 dc_p \Delta T. \]

Equating those two energies and solving for \( R_{\text{mirror}} \), I find that

\[ R_{\text{mirror}} = \sqrt{\frac{\rho R_{\text{spot}}^2 dc_p \Delta T}{F_{\text{sun}} \tau}} \]

One last thing to consider here is that wood in the boat is not perfectly dry, do I should use a \( c_p \) somewhere in between that of wood and water, say 2 J g\(^{-1}\)K\(^{-1}\). Plugging in reasonable numbers for the remaining values, I find that \( R_{\text{mirror}} \sim 10 \text{ m} \), roughly the size of the mirrors used in the Keck telescope.

In getting that number, we made a few unrealistic assumptions about the optics. First, it’s unlikely the Greeks were able to craft true parabolas using the technology that they had. Second, it’s not really true that sun’s rays are perfectly parallel - really they are spread out over a small range of angles. To check those assumptions, let’s think about what kind of spot size would be created by a spherical mirror - if it’s smaller than the smearing out due to the motion of the boat, then we’re fine, but otherwise we’ll need to revise the estimate. Using the method from the laser pointer problem, the spot size will be given by

\[ R_{\text{spot}} = D\theta_{\text{sun}} \]

Where \( D \) is the “distance of a bow shot” - maybe 100 m - and \( \theta_{\text{sun}} \sim 5 \cdot 10^{-3} \) radians is the angular size of the sun. But that says that \( R_{\text{spot}} \sim 50 \text{ cm} \), which is what we were using anyway. So, the two effects are comparable, and even if the Greeks could only use a spherical mirror, they still couldn’t do any worse than the estimate above.

(b) Suppose Archimedes wanted to use instead the burnished, flat, bronze shields of Greek soldiers. Estimate the required number of shields.

The difference is that now the spot size is set by how much the light reflecting off a plane mirror has spread out by a distance of 100 m. I seem to recall from pictures that these Greek soldiers carried rather larger shields, maybe about 1 m tall, since you’d want the shield to protect most of your upper body. That means that if the rays were coming in perfectly parallel, you’d get a spot size of about 1 m. However, with an angular spread of \( \theta_{\text{sun}} \) instead, you’ll pick up an additional \( 2D\theta_{\text{sun}} \), from the rule that for mirrors the angle of incidence equals the angle of reflection (see figure).
That means that for $D = 100\, \text{m}$, the spot will have double in size from 1m to 2m in size. That’s greater than the vertical movement of the boat due to waves, so let’s ignore that effect this time. Following the same steps as above, the energy imparted to the hull of the boat is now

$$nF_{\text{sun}}A_{\text{mirror}}\tau$$

where $n$ is the number of shields. Setting that equal to the energy needed to start a fire, I obtain

$$n = \frac{\rho c_p \Delta T}{F_{\text{sun}} \tau} \left( \frac{R_{\text{shield}}}{R_{\text{spot}}} \right)^2 \approx 1000$$

So it would take on order $1000$ shields to pull this off. The level of concentration required here I think is pretty prohibitive - you’d have to get 1000 people to focus the light on the same spot for ten seconds, while the boat is possibly moving around. While this weapon is physically possible, it’s hard to avoid the conclusion that there are more reliable ways of setting a boat on fire.

4 Cavendish’s Torsional Balance

See the description of Cavendish’s celebrated experimental measurement of the force of gravity at en.wikipedia.org/wiki/Cavendish_experiment.

What the wiki page does not tell you is the radius of the wire used in Cavendish’s torsional balance. Deduce this radius using order-of-magnitude physics and the experimental parameters provided on this wiki page. Provide both a symbolic answer and a numerical estimate.
Hint: Begin by understanding the torsional stresses on the wire. The top cross-sectional face of the wire is cemented to the top of the apparatus, while the bottom face is free to rotate. Draw a line from the top of the wire to the bottom, parallel to the long axis of the wire. In the absence of a torque, this line is straight. In the presence of a torque, this line is bent.

From the description on the Wikipedia page, the Cavendish apparatus consists of two small bobs attached via a wooden arm and suspended from a thin wire with two larger bobs positioned next to the small ones. The gravitational attraction between the bobs causes the arm to rotate, which makes the wire twist. The rotation stops when the restoring torque from the torsion in the wire cancels out the gravitational torque, and by measuring the angle at which that happens you can back out the gravitational force between the bobs. We are asked to figure out the thickness of the wire used.

This problem was all about drawing a good picture. In the figure above, I’ve shown side and cross-sectional views of the wire as it twists. You can see that as a wire of length L and radius R rotates through angle φ, the material undergoes a shear of

$$\theta = \frac{\delta}{L} = \frac{\phi R}{L}$$

Hence, if M is the modulus of the wire, the stress is given by

$$\sigma = M\theta = M\phi \frac{R}{L}$$

and the restoring torque is

$$T = FR = (\sigma \pi R^2)R = \frac{\pi M\phi R^4}{L}$$
That must balance out the torque due to gravity, so

\[
\frac{\pi M \phi R^4}{L} = F_G r = \frac{2G m_1 m_2 r}{d^2} \rightarrow R \sim \left( \frac{2L G m_1 m_2 r}{\pi M \phi d^2} \right)^{1/4}
\]

where \( m_1, m_2, \) and \( d \) are the masses and separation of the bobs and \( r \) is their moment arm, i.e. \( \frac{1}{2} \) of the length of the wooden stick. Plugging in the numbers from the Wikipedia page, I get that \( R \sim 1 \text{ mm} \).