Astro 162 – Planetary Astrophysics – Solutions to Problem Set 2

Problem 1. *Protostellar Disk Sizes*

The gravitational collapse of a molecular cloud to form a star leads inevitably to the formation of a disk that carries most of the angular momentum of the initial cloud. In this problem we will estimate roughly the sizes of these protostellar disks.

Consider a Jeans-unstable clump of molecular gas of mass $M = 1M_\odot$*, radius* $R = 0.1\,\mathrm{pc}$*, and spin rate* $1\,\mathrm{km\,s^{-1}\,pc^{-1}}$*. (Clumps like this have been observed via millimeter-wave observations.) As this cloud collapses inward to form a star, the outer radius of the cloud will decrease. The outer radius will not decrease indefinitely, however: it will stall at the centrifugal barrier. Estimate the radius of the centrifugal barrier in AUs, and compare your answer to the sizes of protostellar disks displayed in class.*

Consider a parcel of gas at the equator of the spinning cloud. This parcel of gas has the greatest angular momentum per unit mass than any other parcel of gas in the cloud. As the cloud (and the parcel) collapse inwards, the angular momentum of this parcel is conserved. The angular momentum of the parcel just prior to collapse equals the angular momentum of the parcel when it hits the centrifugal barrier. The initial specific angular momentum ("specific" means per unit mass) of the parcel is:

$$\begin{aligned} l_i &= \omega_i R_i^2 && (1) \\ &= (1\,\mathrm{km\,s^{-1}\,pc^{-1}})(0.1\,\mathrm{pc})^2 && (2) \end{aligned}$$

The final specific angular momentum of the parcel at the turn-around radius (centrifugal barrier) is

$$\begin{aligned} l_f &= \omega_f R_f^2 && (3) \\ &\approx \sqrt{GM/R_f^3}R_f^2 && (4) \\ &\approx \sqrt{GMR_f} && (5) \end{aligned}$$

where for the second line we have noted that the spin rate of the collapsed cloud should be of order the Keplerian orbital frequency around the entire cloud mass. This is reasonable because all of the extra spin velocity that the collapsed cloud now posesses comes from the gravitational self-attraction of the cloud. The parcel is on the outside of the cloud, so it feels the entire gravitational force from the cloud as if the cloud were a point mass.

Set $l_f = l_i$ and solve for $\boxed{R_f = \omega_i^2 R_i^4/(GM) = 4000\,\mathrm{AU}}$. This is a bit larger than the sizes of protostellar disks displayed in class, but is within an order of magnitude. Remember that astronomical image sizes all depend on optical depth and the wavelengths at which you observe; this problem derives a dynamical radius which is independent of such considerations.

Alternatively, we can estimate that as our parcel falls in, all of the gas interior to it stays interior and remains close enough to spherically distributed that the parcel follows an elliptical Keplerian orbit around the cloud mass M. Our parcel begins at the apocenter of the orbit (the furthest distance from the center that it reaches in the orbit), and when it reaches its pericenter (the closest distance to the center), it collides with other parcels of gas, dissipates energy, and settles down into a circular orbit at about the distance of its pericenter. So, to calculate the parcel's final distance from the center of the cloud, we need to calculate the pericenter distance of its orbit.

If the semi-major axis of the parcel's orbit is a and its eccentricity is e, then the initial radius of the cloud is $R = R_i = a(1+e)$, and the final radius is $R_f = a(1-e)$. We know from equating the energy and momentum at pericenter and apocenter in a Keplerian orbit that the velocity at apocenter is given by $v_{apo} = \omega_{Kep} a\sqrt{(1-e)/(1+e)}$, where ω_{Kep} is the mean Keplerian angular velocity of the orbit. The actual starting velocity of our parcel is $v_i = \omega_i R_i$. Setting $v_{apo} = v_i$, we find $R_f = a(1-e) = \omega_i^2 R_i^3 a/(GM) = \omega_i^2 R_i^4/(GM)/(1+e)$, where we have used Keplers law, $GM = \omega_{Kep}^2 a^3$. This result for R_f is the same as that calculated above divided by $(1+e)$. To calculate e, rewrite our velocity equation, $v_{apo} = v_i$, as $1 - e = R_i v_i^2/(GM)$. Solving for e and plugging in $v_i = \omega_i R_i$, we find $e = 1-(R_i^3\omega_i^2)/(GM)$. For our values $e = 0.8$, so we get $R_f \approx 2000\,\mathrm{AU}$ which is closer.

Problem 2. *Making Ice Cubes in the Protoplanetary Disk*

(a) Assume that all of the cosmically abundant oxygen in the minimum-mass solar nebula takes the form of water (H_2O) vapor.

What is the mass density [g/cm^3] in water vapor at a stellocentric distance of 5 AU? Use the parameters for the solar nebula described in class.

As stated in class, the mass density of the solar nebula is $\rho \approx 10^{-9}\,\mathrm{g\,cm^{-3}}(r/\,\mathrm{AU})^{-2.8}$. So at $r = 5\,\mathrm{AU}$ (the heliocentric distance to Jupiter), $\rho \approx 1.1 \times 10^{-11}\,\mathrm{g\,cm^{-3}}$. But this is the mass density in everything: hydrogen, helium, metals. To get the mass density in hydrogen alone, multiply by the mass fraction, $X = 0.74$, to get $\rho_H = 8.3 \times 10^{-12}$. The mass density in water vapor is $\rho_{H_2O} = \rho_H \times (n_O/n_H) \times (m_{H_2O}/m_H)$, where n is the number abundance and m is the mass of an individual atom or molecule. From the solar photospheric abundances table, $n_O/n_H = 10^{-3.2}$. And $m_{H_2O}/m_H = 18$. Then $\boxed{\rho_{H_2O} = 9.4 \times 10^{-14}\,\mathrm{g\,cm^{-3}}}$.

(b) Estimate the mean thermal speed of water molecules in the solar nebula at this same stellocentric distance. Give your answer in [km/s].

Plug and chug. At $r = 5AU$, the temperature is about 75 K, using the power law relation given in class. Then $v = \sqrt{kT/m_{H_2O}} = \sqrt{1.4 \times 10^{-16} \times 75/(18 \times 1.7 \times 10^{-24})} = 2.6 \times 10^4\,\mathrm{cm\,s^{-1}}$. So the answer is $\boxed{v \approx 0.18\,\mathrm{km\,s^{-1}}}$.

(c) Consider the growth of an ice particle at 5 AU. Assume that the particle accretes water molecules from the gas phase. Estimate the rate at which its radius increases. Express your answer in units of [cm/yr]. Is this calculation promising or discouraging for growing grains in the solar nebula, given that the lifetime of the solar nebula is estimated to be 10^7 yr*?*

Assume that the each water molecule that collides with the ice particle sticks to it with perfect efficiency. Some of you tried to argue that it won't stick because the gravitational attraction between the molecule and the ice particle is too weak. This is true, but gravity is not what makes small things stick together! Electrostatic forces and chemical bonds make small things stick.

The rate at which mass is accreted by the ice particle of radius r is given by $4\pi r^2 \rho_{H_2O} v$, assuming that the relative speed between the ice particle and the water molecule is dominated by the thermal speed of the molecules. Set this equal to $\dot{m}_{particle} = (d/dt)(4\pi\rho_{ice} r^3/3) = 4\pi\rho_{ice} r^2 \dot{r}$. Solve for $\dot{r}$: $\boxed{\dot{r} = \rho_{H_2O} v/\rho_{ice} = 0.06\,\mathrm{cm\,yr^{-1}}}$. This seems pretty $\boxed{\text{promising}}$ for the growth of small grains in the solar nebula whose lifetime is 10^7 yr, but the maximum size to which grains can grow by accretion from the vapor phase depends also on the number of seed nucleation particles you start with, which nobody can estimate confidently right now. Also, our estimate assumes that each grain has no competition from other grains for the gas molecules. In reality, the gas is being eaten up by many grains simultaneously, and this needs to be taken into account.

Problem 3. *Make like a tree and ...*

If all the leaves of a tree fall to the ground, how thick is the layer of leaves on the ground? Give your answer in leaf thicknesses, to order-of-magnitude.

You are lying below the tree. When you look up, the tree presents a certain optical depth to visible light photons coming from the sun. What is the order-of-magnitude of this optical depth? Why does this answer make sense from the perspective of the vegetation? (Be the tree.)

The above two paragraphs are flip sides of the same question. You can try the second paragraph before the first, or vice versa.

The following answer is from Berkeley physics graduate student, Roger O'Brient.

For simplicity, I'll consider trees with leaves that bunch into vertical cylinders. Let's say these trees have a height $s = 10m$. Each leaf should have roughly the surface area of my driver's license ($\sigma \approx 3in \times 2in \approx 5in^2 \approx 30cm^2$). Looking outside, I guess that trees have between 10 and 100 $\frac{leaves}{m^3}$. So I'll split the difference and say that there are $5 \times 10^{-5} \frac{leaves}{cm^3}$.

This means that the optical depth (assuming constant density) is:

$$\boxed{\tau = n\sigma s = (5 \times 10^{-5})(30)(1000) = 1.5}.$$

This makes sense. A tree ought to absorb most of the light falling on it (i.e. $\tau \approx 1$). If it absorbed any less, it could grow leaves near the base to absorb the extra light. And if it absorbed more, then leaves near the base would die due to lack of light exposure. In fact τ probably determines the density of leaves.

As for the part about the depth of leaves on the ground, consider a 10 m tall square column of leaves with a cross sectional area of 5 in^2 that all fall off the tree and collect directly on top of each other on the ground. With the same density as guessed above, there should be:

$$\boxed{nAh = (50\ \tfrac{leaves}{m^3})(3 \times 10^{-3} m^2)(10m) = 1.5 \text{ leaves}}.$$

So the pile is between 1 and 2 leaves thick. Gee-wiz! That's the same as the first part! The questions are mathematically identical!

Problem 4. *Baby Boom*

Landfills are filled with our garbage. In the United States, what fraction of landfills is filled with disposable diapers?

We estimate that:

1. A baby needs about 10 diapers per day.
2. About 1 in 15 households in the US have a baby.
3. An average household throws out 2 large trash bags of trash per week.
4. About half of the trash in landfills is commercial and about half is residential.

Ten diapers per day translates to 70 diapers per week, which will fill up about two large trash bags. So, the fraction of residential trash that is disposable diapers is: (2 bags of diapers)/(15×2 bags of other trash + 2 bags of diapers) = 1/16. Since only half of the total trash is residential, the fraction of total trash in diapers is about $1/32 \approx 3$ %.

According to the book, "Numbers," by activist Anita Roddick, the answer is $\boxed{2\%}$.