Planetary Astrophysics – Problem Set 10

Due Thursday Nov 19

1 Hydrostatic Blackbody Disk

Consider a protoplanetary disk in vertical hydrostatic equilibrium

$$\frac{dP}{dz} = -\rho \frac{GM_\star}{a^2} \frac{z}{a} \tag{1}$$

with (ideal) gas pressure P, disk radius a, stellar mass $M_\star$, gravitational constant G, and height above the disk midplane z. In writing the right-hand side, we took the thin-disk limit $z \ll a$. We wrote this equation in class, solved it assuming the gas is vertically isothermal (temperature T is only a function of a and not of z), and found the disk density to have a vertical Gaussian profile with scale height

$$h(a) = \frac{c_\mathrm{s}}{\Omega} = \frac{\sqrt{\frac{kT}{\mu m_\mathrm{H}}}}{\Omega} \tag{2}$$

where k is Boltzmann's constant, μ is the mean molecular weight, m_H is the mass of hydrogen, and $\Omega(a)$ is the Keplerian angular frequency of the disk.

Now if the disk behaves like a blackbody, then in thermal balance

$$\frac{L_\star}{4\pi a^2} \sin\theta = \sigma T^4 \tag{3}$$

where the incident radiation flux from a star of luminosity $L_\star$ (left-hand side) is balanced by cooling from the disk (right-hand side) with Stefan-Boltzmann constant σ. Here θ is the angle at which starlight strikes the disk surface, defined such that $\theta = 90^\circ$ corresponds to starlight illuminating the disk surface at normal incidence.

Suppose the height $H(a)$ at which starlight is absorbed—the disk photosphere—is a fixed number of scale heights, so that $H = Fh$ with constant F. Suppose further that $H \gg R_\star$ (the star can be modeled as a point source), $H \ll a$ and $\theta \ll 1$ (thin-disk approximation).

[10 points] Solve (2) and (3) simultaneously to find how T and H scale with a (you may assume that both T and H are power laws of a but with unknown exponents that you will solve for). Only proportionalities are required; do not worry about coefficients.

The answers were presented in class but without a complete derivation; this problem asks you to supply a derivation.

2 Dispersion vs. Shear-Dominated Relative Velocities

It is essential in planetary dynamics to evaluate how fast bodies (planets, asteroids, spacecraft, ...) are moving relative to each other. In general, we can distinguish two contributions to the relative velocity. The first contribution comes from orbital eccentricities and/or inclinations. When this first contribution dominates, we call the system "dispersion-dominated". Under dispersion-dominated conditions, the relative velocities arise principally because of NON-circular and/or NON-co-planar motions.

But even if all orbits are circular and co-planar, there is a minimum non-zero relative velocity. This minimum relative velocity arises because bodies on concentric co-planar orbits still move at different speeds because they have different semimajor axes. When the contribution to the relative velocity due to different semimajor axes dominates, we call the system "shear-dominated", because the relative velocities are controlled by the background Keplerian shear, given by Kepler's Third Law (the orbital frequency $\Omega = 2\pi/P$*, where* P *is the orbital period, scales with semimajor axis* a *as* $a^{-3/2}$*).*

Parts (a) and (b) describe the dispersion-dominated regime, while part (c) describes the shear-dominated regime.

(a) [3 points] Consider two bodies on co-planar orbits, each of semimajor axis a*. One orbit is circular, while the other has eccentricity* e*. Both bodies are of negligible mass compared to the central stellar mass* $M_\star$*.*

The two orbits cross. At the crossing points, what is the relative velocity u *between the bodies? We are interested here only in the magnitude of this velocity, so give only the absolute value of* u*.*

You may use the following relations describing a Keplerian orbit: the orbital radius

$$r = \frac{a(1-e^2)}{1+e\cos f} \tag{4}$$

the radial velocity

$$\dot{r} = \frac{\Omega a}{\sqrt{1-e^2}}\, e\sin f \tag{5}$$

and the azimuthal velocity

$$r\dot{f} = \frac{\Omega a}{\sqrt{1-e^2}}\,(1+e\cos f) \tag{6}$$

where $\Omega \equiv \sqrt{GM_\star/a^3}$ *is the Keplerian angular frequency (also called the "mean motion") and* f *is the "true anomaly" (angle between periapse and the body's instantaneous azimuth; e.g., when* $f = 0$*, the body is at periapse; and when* $f = \pi$*, the body is at apoapse). Note that* Ω *is constant for constant* a*, while* $\dot{f}$ *varies with* f*. Note also that* $\Omega a = \sqrt{GM_\star/a}$ *is the familiar circular orbital velocity.*

Assume $e \ll 1$ *and express your answer for* u *to first order in* e*. That is, when Taylor expanding in* e*, keep only terms that depend on* e^1 *and drop higher-order terms (those depending on* e^2 *or higher powers of* e*, which are all smaller than* e^1*). Express your answer for* u *in terms of* Ω*,* a*,* e *and whatever other variables or constants you need.*

(b) [3 points] Repeat part (a), but now for two CIRCULAR orbits having the same semimajor axis a and which are mutually inclined by angle i. The two orbits cross each other at what are called "nodes". Find the relative velocity u at either node (again only magnitude is important; do not worry about the sign). Express in terms of i, Ω, and a, working in the limit $i \ll 1$ rad and therefore keeping only terms linear in i — please simplify trigonometric functions accordingly.

ALSO: Combine the situation in part (b) with the situation in part (a): consider one orbit which is circular, and another orbit which is eccentric AND inclined. Write down an approximate expression for the relative velocity u in this combined case and explain your reasoning.

(c) [4 points] Shear-dominated regime: consider now two orbits that are both circular and co-planar, but having slightly different semimajor axes, a and $a + x$, where $x \ll a$. These two orbits don't technically intersect, but every so often the two bodies do get close to each other, within a radial distance x. We call the moment when the bodies get close to each other a "close encounter".

Derive an expression for the relative encounter velocity u in terms of $\Omega(a)$, a, and x. Work to first-order in x; i.e., when Taylor expanding in x, keep only terms that depend on x^1 and drop higher-order (smaller) terms. Take the absolute value of this velocity — we are not interested in direction, only magnitude.

Full credit only if you obtain the precise numerical coefficient. This is NOT an order-of-magnitude problem.

This problem is trickier than it may seem. Naively subtracting the orbital velocity $v(a) = \Omega(a) \times a$ from the other orbital velocity $v(a+x) = \Omega(a+x) \times (a+x)$, with both velocities evaluated in the lab frame, gives the WRONG answer. This naive procedure falls into "Calvin's trap": think about what this procedure would give you if you applied it to two bodies glued to two different radii of a rigidly rotating disk.

To avoid falling into Calvin's trap, first go into the frame rotating with the first body at semimajor axis a. In this frame, the first body is stationary. Now ask, in this rotating frame, how fast the second body is moving.

3 Leftovers: Eat Them or Throw Them Away?

Well, it depends.

(a) [3 points for all questions] A particle orbits a star of mass $M_\star$ with semimajor axis a and negligible eccentricity. What is its orbital velocity v? ALSO, by what minimum amount would the orbital velocity need to increase to eject the particle from the system? Call this extra velocity $\Delta v_{\text{escape,system}}$. An order-of-magnitude answer suffices.

(b) [4 points for all questions] Consider now this same particle, plus a nearby planet of mass M, radius R, and orbital semimajor axis a. The particle and planet do not orbit each other; both orbit the star with semimajor axes that are similar.

Every so often, the particle and planet encounter each other (come within a minimum distance of each other). After multiple encounters, the particle is on an eccentric and *inclined orbit; it achieves a maximum non-circular/non-coplanar velocity* $\max u$ *that is comparable to the escape velocity from the surface of the planet,* $v_{\rm escape,planet}$.[1] *So we have* $u \sim \max u \sim v_{\rm escape,planet}$.

If $\max(u) > \Delta v_{\rm escape,system}$*, then the particle can, in principle, be ejected from the system. If* $\max(u) < \Delta v_{\rm escape,system}$*, then the particle is unlikely to be ejected but remains on an orbit bound to the star, to be continued to be kicked and ultimately accreted by the planet.*

Give an order-of-magnitude expression for $\max(u)$ *in terms of* M*,* R*, and fundamental constants. Then find an order-of-magnitude expression for the orbital semimajor axis* $a_{\rm crit}$ *outside of which ejection is more likely than accretion, in terms of* M*,* R*, and* $M_\star$*. Evaluate* $a_{\rm crit}$ *for an Earth-like planet and for a Jupiter-like planet around a Sun-like star, in units of AUs.*

(c) [3 points] Suppose a planet orbits inside its $a_{\rm crit}$*. In this regime nearby particles are not ejected, but remain on bound star-centered orbits to be repeatedly kicked by the planet. Ultimately, if one waits long enough, these particles will be consumed by the planet.*[2] *Suppose the planet orbits within a disk of small particles, all of which have been kicked by the planet up to* $\max(u)$ *(in random directions). Assuming the system is dispersion-dominated, with eccentricities comparable to inclinations, give an order-of-magnitude expression for the mass accretion rate* $\dot{M}$ *of particles onto the planet, in terms of* R*,* $M_\star$*,* a*, and* Σ*, the mass surface density of the particles. Simplify your expression so that it does not contain* u *or* M*.*

4 A Protoplanet's Reach Should Exceed Its Grasp (Or Else What's a Hill Sphere For?)[3]

We saw in lecture that "accretion is a dish best served cold": the smaller the relative velocities between solid bodies in the disk (the more dynamically "cold" the disk is), the larger are the gravitationally-focussed accretion cross-sections, and the faster bodies merge and grow.

There is, however, a limit to how small the relative velocities, and thus how large accretion cross-sections, can be. That limit is given by the shear-dominated regime (problem 2c). The goal of this problem is to calculate how far away a particle can be from a planet under shear-dominated conditions and still accrete. We will call this maximum impact parameter $x_{\rm crit}$.[4]

[1] Further amplification of u is not feasible because at this point the probability that another encounter increases u is similar to the probability that the particle collides with the planet. This is shown in the review article by Goldreich et al. (2004) reprinted in the Course Reader.

[2] Assuming they are not consumed by the star, a possibility ignored by this problem but which does arise in real life.

[3] Brownie points for those who get the reference.

[4] While a particle that encounters the planet within $x_{\rm crit}$ CAN accrete, it DOES NOT HAVE TO accrete. This is because under shear-dominated conditions, particle trajectories near a planet can be strongly chaotic and frequently DO NOT lead to collisions/accretion. We discuss this in lecture; it is also treated quantitatively in the review article by Goldreich et al. (2004) reprinted in the Course Reader.

(a) [3 points for all questions] Consider a protoplanet of mass M on a circular orbit of semimajor axis a and Keplerian angular frequency Ω around a star of mass $M_\star$. Consider also a small body (particle) on a concentric circular orbit of semimajor axis $a + x$, where $x \ll a$. The set-up is identical to Problem 2c above, and you will need the answer to that problem for the relative encounter velocity u.

A zoom-in of the situation is shown in Figure 1.

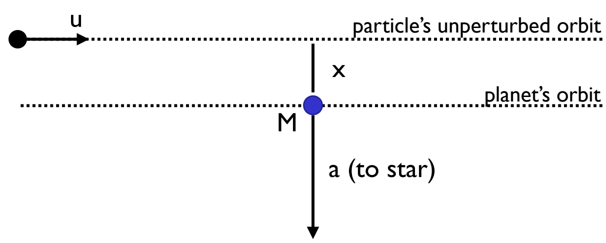

Figure 1: Encounter between a protoplanet and a particle. The impact parameter of the encounter is x, and the relative velocity before the encounter is u.

A "close encounter" between the protoplanet and the particle lasts for as long as the distance between them is roughly the minimum value of $\sim x$. Estimate how long Δt a close encounter lasts. Simplify your answer as much as possible, and express Δt in terms of Ω. ALSO: does Δt depend on x?

(b) [2 points] During the encounter, the test particle's trajectory gets deflected toward the planet. The test particle receives a specific impulse (a "kick" velocity) Δv, directed toward the planet.

Make an order-of-magnitude estimate for Δv in terms of the given variables and fundamental constants. Hint: a velocity is an acceleration multiplied by a time. You may keep order-unity coefficients if you wish, but full credit will be given even if you drop them.

(c) [10 points for all questions] The kick velocity Δv is directed radially inward. Before the kick, the particle was on a circular orbit having zero radial velocity. Just after the kick, the particle finds itself moving inward on a newly eccentric orbit, at an orbital phase of $f \sim 3\pi/2$. The particle has to be near this phase at this moment; it can't be at $f = 0$ or π because the radial velocities are zero at periapse and apoapse, and it can't be at $f \sim \pi/2$ because the radial velocity there is directed radially outward, not inward

(see also Problem 2 above). Thus, just after the kick, the particle is at $f \sim 3\pi/2$*, and it is heading towards its new periapse.*

How far inward is the new periapse away from its original orbit? Call this radial distance Δr *and give an order-of-magnitude expression for it by evaluating the distance traveled at radial speed* Δv *over a quarter of an orbital period, assuming the particle's orbital period has changed negligibly. We take a quarter of an orbital period because it takes roughly that time to move from* $f \sim 3\pi/2$ *to* $f = 2\pi$ *(periapse).*

ALSO: How large must Δr *be in order for the new test particle's orbit to cross the planet's orbit? Solve for the critical value of* $x_{\rm crit}$ *such that orbit crossing—and thus accretion—is just possible. Express* $x_{\rm crit}$ *in terms of* M*,* $M_\star$*, and* a*. FINALLY: compare* $x_{\rm crit}$ *to the Hill radius. Are they the same to order-of-magnitude?*