Planetary Astrophysics – Problem Set 11

Due Thursday Dec 3

1 Hot vs. Cold Meals

(a) [3 points] Show that the Hill sphere radius $R_{\rm H}$ of a planet of mass M, radius R, and bulk density $\rho_{\rm p}$ orbiting a star of mass $M_\star$, $R_\star$, and bulk density $\rho_\star$ at orbital distance a can be written to order-of-magnitude as:

$$R_{\rm H} \sim R/\alpha \tag{1}$$

where

$$\alpha \equiv \left(\frac{\rho_\star}{\rho_{\rm p}}\right)^{1/3} \frac{R_\star}{a} \tag{2}$$

This is a convenient way to estimate $R_{\rm H}$ because typically $(\rho_\star/\rho_{\rm p})^{1/3} \sim 1$. The Hill radius $R_{\rm H}$ is larger than the body radius R by a factor of order $1/\alpha \sim a/R_\star \gg 1$.

(b) [3 points] Show that the orbital period of a test particle orbiting the planet at a separation of $R_{\rm H}$ equals, to order-of-magnitude, the orbital period of the planet orbiting the star at separation a. This is another way to remember $R_{\rm H}$, and lends some intuition as to why a test particle's trajectory near $R_{\rm H}$ is chaotic—it is trying to serve two masters, the planet and the star, simultaneously.

(c) [3 points] The Hill velocity $v_{\rm H} \equiv \Omega R_{\rm H}$ is of order the orbital velocity of a test particle orbiting the planet at a separation of $R_{\rm H}$, where Ω is the angular velocity of the planet around the star ($\sim$ the angular velocity of a test particle around the planet at $R_{\rm H}$, as part (b) shows). The Hill velocity is also the relative velocity of a small body in a shear-dominated disk encountering the planet at an impact parameter equal to $R_{\rm H}$.

Show that $v_{\rm H}$ can be written to order-of-magnitude as $\sim\alpha^{1/2} v_{\rm esc}$, where $v_{\rm esc}$ is the surface escape velocity from the planet.

(d) [3 points] Calculate how much faster shear-dominated accretion is relative to UN-gravitationally focussed dispersion-dominated accretion. Recall from lecture that a planet embedded in a disk of surface density $\Sigma_{\rm solid}$ can, under shear-dominated conditions, accrete at rate $\dot{M}_{\rm shear} \sim \Sigma_{\rm solid}\Omega R_{\rm H}^2 P_{\rm acc}$, where the accretion probability $P_{\rm acc} \sim R v_{\rm esc}/(v_{\rm H} R_{\rm H})$. Recall also that under dispersion-dominated conditions, $\dot{M}_{\rm disp} \sim \Sigma_{\rm solid}\Omega R^2[1+(v_{\rm esc}/v)^2]$.

Derive an order-of-magnitude expression for $\dot{M}_{\rm shear}/\dot{M}_{\rm disp}$, assuming $v > v_{\rm esc}$, in terms of α.

(e) [3 points] Forming the Earth: Consider a proto-Earth of radius $R \sim 4000$ km and bulk density $\rho_{\rm p} \sim 5$ g/cm^3, embedded in a disk of solids of surface density $\Sigma_{\rm solid} \sim 5$ g/cm^2 at an orbital radius $a \sim 1$ AU around the Sun.

Estimate to order-of-magnitude the protoplanet's mass doubling time $M/\dot{M}_{\rm disp}$ assuming dispersion-dominated conditions with $v > v_{\rm esc}$. Express in yr. This is the longest possible time to form the Earth.

(f) [1 point] Estimate to order-of-magnitude the shortest possible mass doubling time $M/\dot{M}_{\rm shear}$ for the proto-Earth assuming shear-dominated conditions. The easiest way to do this problem is to combine (d) and (e). Express in yr.

(g) [3 points] Repeat (e) but for a proto-Neptune of radius $R \sim 15000$ km and bulk density $\rho_{\rm p} \sim 1$ g/cm^3, embedded in a disk of solids of surface density $\Sigma_{\rm solid} \sim 0.1$ g/cm^2 at $a \sim 30$ AU. Assume as in (e) dispersion-dominated conditions with $v > v_{\rm esc}$. Express in yr. Is your answer encouraging or discouraging?

(h) [1 point] Repeat (g) for the proto-Neptune but under shear-dominated conditions. Is this encouraging or discouraging?

2 To Cool is to Accrete

A solid core embedded in a gaseous circumstellar disk can accrete disk gas. The accreted gas forms a proto-atmosphere around the planet. The proto-atmosphere can continue to gain mass from the surrounding disk by cooling (radiating its energy into space). Cooling contracts the atmosphere (its scale height shrinks) and allows fresh gas from the disk at large to re-fill the planet's atmospheric volume. To cool is to accrete:

$$\frac{E}{L} \sim \frac{M_{\rm gas}}{\dot{M}_{\rm gas}}\,. \tag{3}$$

The left-hand side is the cooling time of the atmosphere (a.k.a. the Kelvin-Helmholtz time), where E is the thermal energy of the atmosphere and L is the luminosity (power radiated into space by the atmosphere). The right-hand side is the mass-doubling time of the proto-atmosphere of mass $M_{\rm gas}$ accreting at rate $\dot{M}_{\rm gas}$. Cooling-limited accretion is thought to be relevant for super-Earth atmospheres and for the initial growth of giant planet (Jupiter-class) atmospheres (e.g., Pollack et al. 1996; Lee & Chiang 2015).

Equation (3) should look familiar: we encountered it before when we derived planet cooling curves. The difference between the cooling problem we treated earlier and the cooling problem we are examining here is that the former problem considered a planet cooling into the vacuum of space at fixed mass, whereas the current problem considers a planet cooling into its natal nebula and accreting more mass from the nebula as a consequence of that cooling.

We have seen in this course that many planetary atmospheres are predominantly convective. Proto-atmospheres are no exception. Consider a convecting proto-atmosphere of mass $M_{\rm gas}$ atop a solid core of radius $R_{\rm core}$ and mass $M_{\rm core} \gg M_{\rm gas}$. Take the atmosphere to behave as an ideal gas and to have an adiabatic index (usual ratio of specific *heats) equal to γ. The atmosphere extends from $R_{\rm core}$ to an outer radius $R_{\rm rcb}$ (radius of the radiative-convective boundary). At the rcb, the temperature equals $T_{\rm rcb}$ and the mass density equals $\rho_{\rm rcb}$.*

(a) [5 points] Starting from hydrostatic equilibrium (with $g = GM_{\rm core}/r^2$—remember that $M_{\rm core} \gg M_{\rm gas}$ and therefore the self-gravity of the atmosphere can be neglected), and assuming an adiabatic temperature gradient, derive the atmospheric mass density profile:

$$\rho = \rho_{\rm rcb}\left[1 + \nabla_{\rm ad}\frac{GM_{\rm core}}{c_{\rm rcb}^2}\left(\frac{1}{r} - \frac{1}{R_{\rm rcb}}\right)\right]^{1/(\gamma-1)} \tag{4}$$

where $c_{\rm rcb}^2 \equiv kT_{\rm rcb}/(\mu m_{\rm H})$ and $\nabla_{\rm ad} \equiv d\ln T/d\ln P = (\gamma-1)/\gamma$ (i.e., $T \propto P^{(\gamma-1)/\gamma}$ for an ideal gas that behaves adiabatically, where P is gas pressure). The other variables G, k, μ, and $m_{\rm H}$ have their usual meanings.

(b) [5 points for all questions in this part] Now since $1/r > 1/R_{\rm rcb}$ and $GM_{\rm core}/(c_{\rm rcb}^2 r) > GM_{\rm core}/(c_{\rm rcb}^2 R_{\rm rcb}) \sim 1$ (this last equality follows from hydrostatic equilbrium at $R_{\rm rcb}$—try justifying it to yourself if you want), we can simplify (4) as:

$$\rho \sim \rho_{\rm rcb}\left(\nabla_{\rm ad}\frac{GM_{\rm core}}{c_{\rm rcb}^2 r}\right)^{1/(\gamma-1)} \tag{5}$$

Now by definition

$$M_{\rm gas} = 4\pi\int_{R_{\rm core}}^{R_{\rm rcb}} r^2\rho(r)\,dr \tag{6}$$

For $\gamma < \gamma_{\rm crit}$, the integral above cares more about the lower limit than the upper limit. What is $\gamma_{\rm crit}$? For $\gamma < \gamma_{\rm crit}$ we say the atmosphere is centrally concentrated—most of its mass is concentrated toward the core.

Detailed models of the thermodynamics of super-Earth atmospheres reveal that $\gamma < \gamma_{\rm crit}$ for much of the atmosphere—dissociation of H_2 renders the gas nearly isothermal with increasing depth (for a similar reason the temperature of a water-ice mixture stays nearly constant as the ice melts—energy is going into breaking bonds but not raising the kinetic energy). The more isothermal an atmosphere is, the closer γ is to its minimum value of 1.

Assuming the lower limit in (5) dominates the integral (i.e., assuming $\gamma < \gamma_{\rm crit}$), write down how $M_{\rm gas}$ scales with $R_{\rm core}$, $M_{\rm core}$, $T_{\rm rcb}$, and $\rho_{\rm rcb}$. All that is required is a proportionality; ignore coefficients. Your answer should contain γ in some of the exponents.

(c) [5 points for all questions in this part] Equation (5) is a power law for ρ in r. In general, a power law $y \propto x^\alpha$ is "scale-free" in the sense that the scale over which y changes by an order-unity factor (say 2) is just given by the local x,[1] whatever that is. Thus, for example, at $r \sim R_{\rm core}$, ρ changes by a factor of 2 over a length scale of order $r \sim R_{\rm core}$; and at a much larger radius, say $r \sim R_{\rm rcb}$, the density changes by a factor of 2 over a length scale of order $r \sim R_{\rm rcb}$. We say the behavior on small scales is similar to the behavior on large scales ("self-similar").

[1] Assuming α is also of order unity and not something crazy like 20.

Given our finding in (b) that the atmosphere is centrally concentrated toward the core, we have $E \sim [M_{\rm gas}/(\mu m_{\rm H})]kT_{\rm c}$, where $T_{\rm c}$ is the (central) temperature of the gas at $r \sim R_{\rm core}$.

From hydrostatic equilibrium near $r \sim R_{\rm core}$, and the power-law, scale-free nature of ρ (and by extension P and T since all these variables are just powers of one another), show to order-of-magnitude that $kT_{\rm c} \sim GM_{\rm core}\mu m_{\rm H}/R_{\rm core}$.

Thereby show that $E \sim GM_{\rm core}M_{\rm gas}/R_{\rm core}$, which just says the thermal energy equals the absolute magnitude of the gravitational potential energy—a statement of virial equilibrium.

(d) [5 points] The luminosity L is controlled by the radiative-convective boundary (rcb). The rcb acts as a "lid" that controls how much heat gets out because it's there that the temperature gradient dT/dr is steepest and the radiative flux greatest (since radiative flux is proportional to dT/dr). The temperature gradient in the radiative layer cannot be any steeper than at the rcb; if it were steeper, then convection would be triggered and the rcb wouldn't be where it is. Thus the flux of energy transported through the radiative layer is throttled at its base, at the rcb.

At the rcb, energy transport is equal parts radiative diffusion and convection (convection below is ceding its dominance to radiation above). We can approximate the temperature gradient dT/dr at the rcb using either radiative diffusion, here expressed using Fick's Law:

$$F = \frac{L}{4\pi r^2} \sim -D\nabla u \tag{7}$$

where $u = aT^4$ is the energy density of blackbody radiation, a is the radiation constant, and D is the diffusivity. Or we can take the temperature gradient from the adiabat; recall that in a convecting atmosphere, the temperature gradient dT/dr can be approximated as adiabatic. We do not give the formula for $(dT/dr)_{\rm ad}$ here; you were asked to derive it in an earlier problem set, and we also gave it in class.

Set the radiative temperature gradient as given by (7) equal to the adiabatic temperature gradient to find how L at $r = R_{\rm rcb}$ scales with $M_{\rm core}$, $T_{\rm rcb}$, $\kappa_{\rm rcb}$, and $\rho_{\rm rcb}$, where $\kappa_{\rm rcb}$ is the opacity at the rcb. All that is required is a proportionality; ignore coefficients. Use relations we learned about earlier in the course.

(e) [5 points] Suppose

$$\kappa \propto \rho^{\alpha} T^{\beta}\,. \tag{8}$$

Use this relation, your answers for (b), (c), and (d), plus $R_{\rm core} \propto M_{\rm core}^{1/3}$ (at fixed core density—an OK approximation) to solve for how the atmospheric mass-doubling time $M_{\rm gas}/\dot{M}_{\rm gas}$ scales with $M_{\rm core}$, $M_{\rm gas}$, and $T_{\rm rcb}$ for $\gamma = 1.2$, $\alpha = 0.5$, and $\beta = 1$.

(f) [5 points for all questions in this part] As long as there isn't too much dust in these proto-atmospheres, $T_{\rm rcb}$ more-or-less tracks the background disk temperature, i.e., the outer radiative layer of the atmosphere down to the rcb is approximately isothermal. All

other factors being equal, is it easier to accrete an atmosphere far from the star or close to the star?[2]

Imagine two cores having the same $M_{\rm core}$ and $T_{\rm rcb}$, but one begins with a gas mass $M'_{\rm gas}$ larger than the other $M_{\rm gas}$. What is the ratio of their gas accretion rates $\dot{M}'_{\rm gas}/\dot{M}_{\rm gas}$ in terms of $M'_{\rm gas}/M_{\rm gas}$? As both cores accrete gas, do their gas masses converge or diverge? Qualitatively, is this calculation promising for explaining why so many super-Earths have gas mass fractions that are, to order-of-magnitude, the same (roughly $\sim$1%)?

Finally calculate, for a given $M_{\rm core}$ and $T_{\rm rcb}$, how $M_{\rm gas}$ scales with time t, assuming $M_{\rm gas} = 0$ at $t = 0$. Just a proportionality is sufficient.

3 Getting Your Kicks on Route 162

Consider a particle of mass m encountering a mass $M \gg m$ at relative velocity v and impact parameter b (Figure 1).

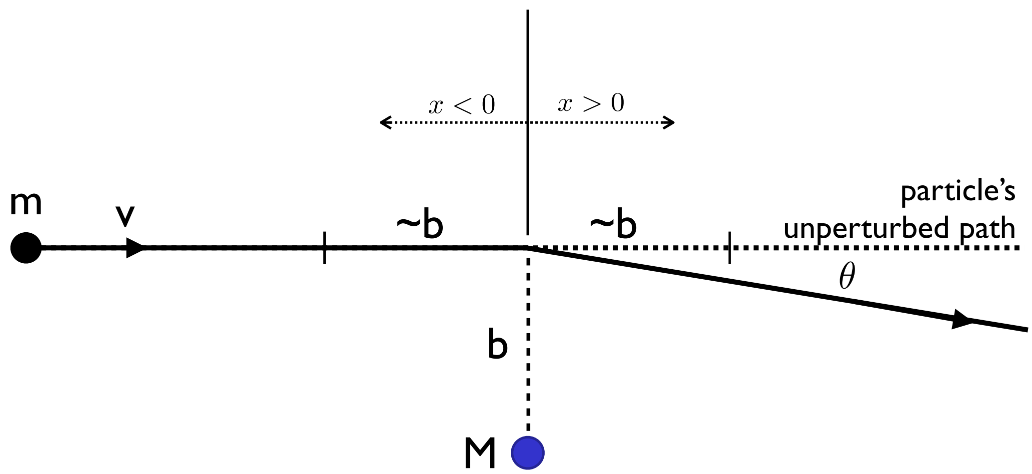

Figure 1: Encounter between m and $M \gg m$. The impact parameter of the encounter is b, and the relative velocity before the encounter is v. The angle of deflection $\theta \ll 1$; the particle velocity v after the encounter is almost (but not exactly) the same as the velocity before the encounter.

(a) [5 points for all questions] The particle receives a kick velocity $\Delta v_\perp$ perpendicular to its original direction of motion. Estimate $\Delta v_\perp$ in terms of M, b, v, and fundamental constants. You may use the impulse approximation.

[2] Although some would say we need more data, exoplanet observations so far indicate that, in a gross average sense, characteristic planet gas masses increase with increasing disk radius, from $\sim$0.01 AU out to about 5–10 AU. That is, gas giants tend to be at large distances from their host star, and rocky bodies tend to be at small distances. Certainly the Solar System fits this trend. This problem is offering one reason why this might be the case (there may be other reasons).

Estimate, to order-of-magnitude, the deflection angle θ, in terms of M, b, v, and fundamental constants.

Assume throughout that $\theta \ll 1$.

For this part (a), you may neglect the motion of $M \gg m$.

(b) [5 points] The particle also receives a kick velocity $\Delta v_{\parallel}$ parallel to its original direction of motion. It is much smaller than $\Delta v_{\perp}$, because the interaction of m and M at $x < 0$ (which speeds up the particle) nearly cancels the interaction of m to M at $x > 0$ (which slows down the particle).

The cancellation is not perfect, however, because the downward deflection $\Delta v_{\perp}$ makes the interaction at $x > 0$ slightly stronger than the interaction at $x < 0$. That is, m is slightly closer to M at $x > 0$ than at $x < 0$. See Figure 2.

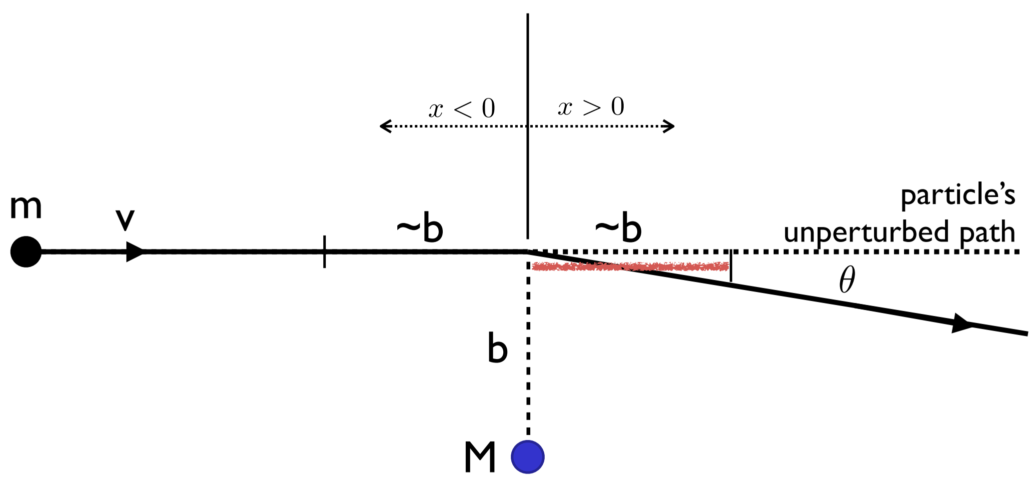

Figure 2: The interaction at $x > 0$ is a little stronger than the interaction at $x < 0$. That is, m is a little closer to M at $x > 0$ because of the downward deflection $\Delta v_{\perp}$. The red line roughly approximates how much closer m is to M during the latter half of the encounter at $x > 0$. The slight vertical offset between the red line of length ∼b at $x > 0$ and the black line of equal length ∼b at $x < 0$ gives rise to an imperfect cancellation and thus a non-zero kick $\Delta v_{\parallel}$.

Estimate $\Delta v_{\parallel}$ in terms of $\Delta v_{\perp}$ and θ, assuming $\theta \ll 1$. Simplify your expression as much as possible (hint: you can Taylor expand). You may use Figure 2 and its caption for inspiration. Be sure to give the sign of $\Delta v_{\parallel}$ (positive in the positive x-direction, negative in the negative x-direction).

For this part (b) you may continue to neglect the motion of $M \gg m$.

(c) [7 points] Now imagine M and m are initially both on circular orbits around a

star before the encounter, with m on an orbit slightly larger than M. This is a shear-dominated situation, with b much larger than the Hill sphere of M.[3]

Does m gain or lose orbital angular momentum after the encounter? Explain all steps in your reasoning — draw pictures! (Hint: Does the relative velocity between m and M decrease or increase after the encounter?)

Thus decide whether m's new semimajor axis after the encounter is larger or smaller.[4]

This is the basis of gap formation in planetary rings and protoplanetary disks.

(d) [3 points] Now we relax the assumption that M does not move. Momentum is conserved in this two-body encounter, so whatever momentum is gained by m is lost by M. Thus in the same orbital set-up of part (c), decide whether M's new semimajor axis after the encounter is larger or smaller.

This is the basis of planetary migration.

[3] As a result, the encounter does not result in orbit crossing. See a previous problem set.

[4] Recall a previous problem set where we examined the eccentricity of the test particle after the encounter. That previous problem also assumed the change in orbital period after the encounter was negligible. The present problem is asking you to examine, qualitatively, the change in semimajor axis, which in turn changes the orbital period. These changes in semimajor axis and period tend to be smaller than the change in eccentricity ($\Delta a/a \ll \Delta e$) but they are not zero, and over time they can lead to dramatic changes in orbital evolution.