Planetary Astrophysics – Problem Set 2

Due Thursday Sep 17

1 The Incredible Invariant Radius

[10 points for all parts]

(a) Combine the equation for hydrostatic equilibrium,

$$\nabla P = \rho \vec{g}\,, \tag{1}$$

Poisson's equation for gravity,

$$\nabla^2 \phi = 4\pi G \rho\,, \tag{2}$$

and an approximate equation of state for liquid hydrogen,

$$P = K\rho^2\,, \tag{3}$$

to derive a single second-order differential equation for density as a function of radius, $\rho(r)$. Here ϕ is the gravitational potential, $\vec{g} = -\nabla\phi$ is the gravitational acceleration, P is pressure, and K is a constant.

Notice in the equation for hydrostatic equilibrium as we have written it, there is no minus sign, unlike what we wrote in class. This is because we are writing here the vector form of hydrostatic equilibrium, whereas in class we were using the scalar form (where g the scalar was assumed to be positive; in other words, $\vec{g} = -g\hat{r}$, where $\hat{r}$ points in the radial direction).

Assume spherical symmetry—so work in spherical coordinates. For spherically symmetric objects,

$$\nabla F = \frac{dF}{dr}\hat{r} \tag{4}$$

and

$$\nabla^2 F = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dF}{dr}\right) \tag{5}$$

for some function $F(r)$, and $\hat{r}$ is the unit vector pointing in the radial direction.

This problem is yet another exercise in hydrostatic equilibrium, except that we now include self-gravity and do not assume that the gravitational acceleration is constant (unlike in the liquid giant problem). The governing ordinary differential equation for

$\rho(r)$ that you will derive is called the Lane-Emden equation, and its solution is a so-called polytrope: a self-gravitating, hydrostatic fluid in which P is assumed to be a power law of ρ. The solutions for $\rho(r)$ typically pass through zero (i.e., they go from positive to negative) at some finite radius; we cut off the solution there and declare this zero-crossing to be the radius of the object.

Polytropes are useful models for stars and planets. In only a few cases is the solution analytic, and this problem examines one of them.

(b) The solution to the equation you have derived (assuming you have the right one; at least check your units) is

$$\frac{\rho}{\rho_0} = \frac{\sin(\pi r/R)}{(\pi r/R)} \tag{6}$$

where R is the outer radius of the body, since ρ vanishes at $r = R$. Derive, using (a) and the equation above, a symbolic expression for R in terms of the variables given and fundamental constants.

(c) Numerically evaluate R for Saturn, Jupiter, and a brown dwarf having 50 Jupiter masses, using $K = 2.7 \times 10^{12}$ [cgs].[1]

2 A License to Fuse

(a) [4 points] Derive an order-of-magnitude analytic expression for the radius $R_{\rm deg}$ of an electron-degenerate brown dwarf in terms of its mass M and fundamental constants. Use the order-of-magnitude relations we discussed in class and omit all numerical coefficients.[2] *Assume the brown dwarf is composed purely of hydrogen.*

(b) [4 points] Derive an order-of-magnitude analytic expression for the radius $R_{\rm ideal}$ of a star supported by ideal gas pressure, in terms of its mass M, its central temperature $T_{\rm c}$, and fundamental constants. Assume the star is composed purely of hydrogen. Again discard all numerical coefficients.

Hydrogen burning on the main sequence requires $T_{\rm c} \sim 10^7$ K (see your stars course to understand why).

(c) [2 points] Consider an object of mass M that cools and contracts under gravity during its early evolution. Initially it is diffuse and acts as an ideal gas. In contracting it is trying to establish hydrostatic equilibrium — to attain a radius R such that its internal pressure balances gravity. If $R_{\rm deg} > R_{\rm ideal}$, where $R_{\rm ideal}$ is evaluated for $T_{\rm c} \sim 10^7$ K, then the gas becomes degenerate first and stabilizes before burning hydrogen—it is a brown dwarf (a "failed star"). If instead $R_{\rm ideal} > R_{\rm deg}$, then the gas first starts burning hydrogen and stabilizes that way—it is a star (whose interior can still be pretty degenerate, as M dwarf cores tend to be).

[1] K is a measure of the entropy per unit mass of the object, as we will discuss in class.

[2] Because who are we kidding?

From this reasoning derive an analytic order-of-magnitude expression for $M_{\rm crit}$, the mass dividing stars that stably fuse hydrogen, from sub-stellar objects (brown dwarfs and planets) that don't. Express your answer in terms of fundamental constants and $T_{\rm c}$, and also evaluate numerically in units of $M_{\rm J}$.

(You may compare your estimate to the canonical $M_{\rm crit} \approx 80 M_{\rm J}$.)

3 Brown Dwarfs and White Dwarfs

What's the difference between a brown dwarf and a white dwarf?[3]

[5 points] Structurally, brown dwarfs and white dwarfs are similar insofar as both are supported by free electron degeneracy pressure against gravity. In class, we derived how radius R scales with mass M for such objects. Extend the derivation to decide how radius scales with M, the atomic mass number A, and the number of electrons per nucleus η_e. Use this scaling relation to estimate the ratio of brown dwarf to white dwarf radii, $R_{\rm BD}/R_{\rm WD}$ (at zero temperature). Consider a brown dwarf of mass $50 M_{\rm J}$ composed purely of ionized hydrogen, and a white dwarf of mass $0.5 M_\odot$ composed of fully ionized carbon and oxygen.

(You can compare your estimate to the actual ratio of $R_{\rm BD}/R_{\rm WD} \approx 1 R_{\rm J}/1 R_\oplus \approx 10$.)

4 Helium Rain

Saturn's total luminosity is about 1.7 times greater than the power that it absorbs from sunlight. It is thought that the gravitational settling of helium, out of the fluid envelope of Saturn onto the core of Saturn, is responsible for this extra power. The pressure in the interior of Saturn is high enough that helium is a liquid. It is a liquid that is immiscible (doesn't mix) with hydrogen, which is also liquid. The liquid helium literally rains to greater depths within the planet, converting its gravitational potential energy to heat.

(a) [10 points] If all of the helium from Saturn's envelope rains onto the surface of the rocky core of Saturn, how much energy would be released? Answer in [ergs]. An order-of-magnitude answer is sufficient; you don't need to use any complicated model for Saturn's interior structure. Take Saturn, which has a total mass of $100 M_\oplus$, to have a rocky core having a mass of $m_{\rm core} \sim 10 M_\oplus$ and a mean density of $\rho_{\rm core} = 7$ g/cc, and its outer liquid envelope to be of solar composition, so composed of 75% hydrogen and 25% helium by mass. The helium is initially well-mixed with the hydrogen; after rain-out, the helium has precipitated out and been absorbed onto the surface of the rocky core.

Use whatever assumptions and approximations you need to, but please be neat and clear about them. One simplification that I took is to assume the envelope to be of uniform density.

[3] Aside from the color. To the human eye, brown dwarfs would be appear magenta as a consequence of pressure-broadened absorption bands from the alkali metals Na and K, while young white dwarfs, whose luminosities peak in the ultraviolet at effective temperatures $T \sim 10^5$ K, appear blue-white. I think the "brown" in "brown dwarf" is supposed to connote an electrical "brown-out"; the inability to fuse hydrogen means the lights are out.

(b) [3 points] Using the fact that Saturn radiates 1.7 times as much as it absorbs from sunlight, calculate how much EXCESS energy is released by Saturn over the age of the solar system ($t = 4.6 \times 10^9$ yr), assuming its luminosity has been constant for this time. Take Saturn's so-called Bond albedo (the fraction of sunlight that is not absorbed but scattered back into space–from ammonia clouds) to be $A = 0.34$.

(c) [2 points] Compare (a) to (b) and comment on whether helium precipitation is a good candidate for Saturn's excess luminosity.