Planetary Astrophysics – Problem Set 4

Due Thursday Oct 1

1 Sub-Neptunes Everywhere

Sub-Neptune exoplanets, found ubiquitously at stellocentric distances of $\sim$*0.1–1 AU by the Kepler transit survey, have masses of* $M \sim 5$–$10 M_\oplus$ *and radii* $2R_\oplus \lesssim R \lesssim 4R_\oplus$*. The going interpretation is that these planets have solid cores that dominate their mass (so core mass* $M_{\rm core} = M$*) and that these cores are overlaid by gaseous, hydrogen-rich envelopes having a fractional mass* $f \sim 0.001$–0.1*. These gas envelopes, though modest in mass, are substantial in volume—detailed models reveal that such envelopes can increase the radius of the planet by a factor of a few, from the solid core radius of* $R_{\rm core} \approx 1.6 R_\oplus$ *to the observed transit radius* $2R_\oplus \lesssim R \lesssim 4R_\oplus$*. This problem tries to reproduce this radius enhancement, and along the way provides some intuition about atmospheres, and some practice with optical depth.*

(a) [10 points] The gas envelope divides into a convective interior and a radiative exterior. First consider the convective interior. Take the convective interior's pressure P *and mass density* ρ *to obey a simple power-law adiabat (we will talk about what adiabats mean later in the course):*

$$P = P_{\rm rcb}\,(\rho/\rho_{\rm rcb})^{\gamma} \tag{1}$$

where "rcb" denotes the "radiative-convective boundary", located at radius $R_{\rm rcb}$*, between the convective zone and the radiative zone. This expression is only good for* $\rho > \rho_{\rm rcb}$ *at* $r < R_{\rm rcb}$ *(since* ρ *increases as one descends from the rcb into the convective zone).*

Insert the adiabat into the equation of hydrostatic equilibrium to derive $\rho(r)$. *Since* $f \ll 1$*, we may safely assume the envelope's gravity is negligible compared to that of the core (but do not assume the gravitational acceleration* g *is constant). Further assume that the gas behaves ideally (this is a sub-Neptune, not a giant).*

Express your answer in the form $\rho(r) = \rho_{\rm rcb} F(r)$ where $F(r)$ is a (dimensionless) function that depends on $R_{\rm rcb}$ and

$$R_{\rm vir} \equiv \frac{\gamma - 1}{\gamma}\frac{G M \mu m_{\rm H}}{k T_{\rm rcb}} \tag{2}$$

where μ *is the mean molecular weight,* $m_{\rm H}$ *is the mass of the hydrogen atom,* $T_{\rm rcb}$ *is the temperature at the rcb,* G *is the gravitational constant, and* k *is Boltzmann's constant. You may check that your answer satisfies* $F(R_{\rm rcb}) = 1$ *and* $F > 1$ *as* $r < R_{\rm rcb}$*. Note that*

$R_{\rm vir}$ is just a radius defined for mathematical convenience; it doesn't actually correspond to any physical radius in this problem.[1]

(b) [10 points] The gas envelope has mass $M_{\rm env} = fM_{\rm core} = f(4\pi/3)\rho_{\rm core}R_{\rm core}^3$. Assume all of this envelope mass is in the convective zone. There is also some mass in the radiative zone, but assume the radiative mass is much smaller than the convective mass.

Given $f = 0.01$, $R_{\rm core} = 1.6R_\oplus$, $M_{\rm core} = M = 8M_\oplus$ (so $\rho_{\rm core} = 10.7$ g/cm^3), $P_{\rm rcb} = 100$ bars, $T_{\rm rcb} = 2000$ K, $\mu = 2$, and $\gamma = 3/2$, **solve for $R_{\rm rcb}$ in terms of $R_{\rm core}$.**[2]

This is a matter of integrating your answer in (a) (multiplied by the appropriate variables) from $r = R_{\rm core}$ to $r = R_{\rm rcb}$ to satisfy $M_{\rm env}$. We leave the writing of the integral to you.

You may solve this problem any way you like. Some may prefer a direct numerical integration. Others may prefer a more analytic approach. If you try for an analytic solution, you may find the following integral helpful:

$$\int x^2 \left[b\left(1/x - 1/a\right) + 1\right]^2 \, dx = \frac{(bx - a(b+x))^3}{3a^2(b-a)} \tag{3}$$

(courtesy Wolfram Alpha). One can simplify the algebra a bit by recognizing that $R_{\rm vir} \gg R_{\rm core}$—if you use this inequality (you don't have to), show it first by numerically evaluating $R_{\rm vir}$ and comparing it to $R_{\rm core}$.

When we did the problem using the above integral (you don't have to—again, you can solve this any way you like), we found a polynomial equation for $R_{\rm rcb}$. One should check that every term in the polynomial has the same units. We solved this polynomial numerically by playing around with a calculator. It was cumbersome to evaluate the polynomial in cgs units (although computers can handle pretty big numbers); we found it a bit more convenient to convert the polynomial into "natural units" by setting $R_{\rm core} = 1$. In these units, $R_{\rm vir}$ is equal to some other pure number (which you should evaluate), and $R_{\rm rcb}$ is still another pure number that, once you find it, will automatically be in units of $R_{\rm core}$, as desired (how nice!).

Full credit awarded for finding $R_{\rm rcb}$ in terms of $R_{\rm core}$ to 2 significant figures, with work and reasoning shown.

Comment (not necessary for the solution): This problem assumes that we know $P_{\rm rcb}$, $T_{\rm rcb}$, and γ. These are not obvious, but arise from detailed modeling (e.g., Lopez & Fortney 2014; Lee, Chiang & Ormel 2014). Still, we can make some general remarks. The temperature $T_{\rm rcb} \sim 2000$ K marks the dissociation front for molecular hydrogen; the dissociation of H_2 into H also leads, as a byproduct, to H^- (the extra electron coming from ionized trace metals), and H^- has a large broadband opacity which forces convection (H^- is also what makes the Sun's surface opaque at visible wavelengths). A dissociation temperature of ∼2000 K follows from the Saha equation (applied to hydrogen dissociation, not hydrogen ionization). By extension, although in detailed interior

[1] Although theoretical physicists who like abstractions will recognize that $R_{\rm vir}$ is a kind of "virial" radius.

[2] For fun and personal edification you may calculate how $R_{\rm rcb}$ varies as f varies from 0.001 to 0.1. But this is completely optional and no extra credit will be awarded.

models γ is not a constant, an effective, gross-average value of $\gamma = 3/2$ is not crazy, as it sits between 7/5 (appropriate for H_2) and 5/3 (appropriate for H). As for $P_{\rm rcb}$, we can look to Solar System giants for guidance, where $P_{\rm rcb} \sim 1$–10 bars; for close-in sub-Neptunes that are more strongly irradiated by their host stars, $P_{\rm rcb}$ is higher, since stronger stellar radiative heating creates a deeper, nearly isothermal, radiative exterior.

(c) [10 points] Now we examine the radiative exterior. Assume it is isothermal at a mean temperature of $T_{\rm rad}$ (the subscript rad denotes the radiative zone). **Derive an analytic expression for $\rho(r > R_{\rm rcb})$ in terms of $R_{\rm rcb}$, $\rho_{\rm rcb}$, M, the molecular thermal velocity**[3]

$$c_{\rm s} \equiv \sqrt{\frac{kT_{\rm rad}}{\mu m_{\rm H}}} \tag{4}$$

and fundamental constants. *You may check that $\rho(r = R_{\rm rcb}) = \rho_{\rm rcb}$ and that ρ decreases as $r > R_{\rm rcb}$.*

Comment: If you have the correct expression, you will see that as $r \to \infty$, $\rho \to$ constant $\neq 0$ — which seems wrong, as it says that the planet fills the entire universe and has infinite mass. Your expression is OK, though — it's just that in reality, the hydrostatic solution you have written down applies over only a limited interval in radius, and you're not allowed to use it when r goes to ∞. Above a certain height ($r \gtrsim GM/c_{\rm s}^2$, where the thermal velocity exceeds the local escape velocity from the planet), the atmosphere stops being hydrostatic; it becomes hydrodynamic, transitioning into an outflowing wind that causes the planet to lose mass (often, but not always, at a negligibly slow rate). The equations needed to describe the wind are the equations of hydrodynamics (take Astro C202).

(d) *[5 points total for several questions] It is useful to calculate a "local scale height," the distance over which the density (or pressure; density and pressure vary the same way for our assumed isothermal ideal gas) changes by a set amount, say a factor of $e\,(= 2.718...)$:*

$$H \equiv \left(-\frac{d\ln\rho}{dr}\right)^{-1} . \tag{5}$$

Write down an expression for $H(r)$ *in terms of M, $c_{\rm s}^2$, and fundamental constants.* **As r increases, does the scale height H increase or decrease?**

Show further that $H(r)$ may be heuristically[4] **derived by asking what height a particle of mass $\mu m_{\rm H}$ attains if it is launched upward from r, at approximately constant gravitational acceleration g,**[5] **with a kinetic energy equal to $kT_{\rm rad}$.**

[3]More frequently called the "speed of sound" in hydrodynamics, which explains the subscript s in equation 4.

[4]A heuristic argument is one that is not rigorous but that provides some intuition for what is going on. Purists hate heuristics. Personally I can't live without them. The thought experiment of throwing a particle up and seeing how far it travels is heuristic because if we actually did this, the particle would not travel the full distance H, but would collide with a neighboring particle first; the fluid equations that we are using assume that the collisional mean free path $\lambda_{\rm mfp}$ is much less than any macroscopic lengthscale of interest (in this case H).

[5]Here, for this specific sub-problem, the assumption of constant g is valid so long as the height H that you calculate $\ll r$, as r is the scale over which $g = GM/r^2$ changes.

Provide also a numerical evaluation for H/r *at* $r \sim 3R_\oplus$, $M = 8M_\oplus$, $T_{\rm rad} = 500$ *K, and* $\mu = 2$. *If* $H/r \ll 1$, *the atmospheric profile is sharp (the gas density decays exponentially over a distance much less than the overall radius); if* $H/r \gtrsim 1$, *then the atmosphere is more diffuse.*

(e) [5 points total for several questions] The photosphere of an object (planet, star, whatever) is the depth to which you can see from the outside—the depth at which the atmosphere transitions from optically thin to optically thick. The transition is fuzzy (like seeing into the ocean) but is customarily marked by the location where the optical depth

$$\tau \equiv -\int_\infty^r \rho\,\kappa\,dr \tag{6}$$

reaches a value of unity (i.e., 1; we say that $\tau \ll 1$ *is optically thin while* $\tau \gg 1$ *is optically thick). Here* κ *is the opacity = the cross-section for light extinction (absorption and scattering of photons) per unit mass (so* κ *has units of* cm^2/g*). Note that our definition measures optical depth from the outside (other definitions don't necessarily; the convention depends on the problem).*

Given κ, *one can try to perform the integral in (6) to find the* r *at which* $\tau = 1$ *(the photosphere). There are a couple of difficulties with this. The first is that if we use the answer for (c) for* ρ, *then the integral explodes because* ρ *is non-zero at* $r = \infty$ *(see comment under c). The second is that even if we don't start at* ∞ *but replace the lower limit of the integral in (6) with some finite* $r_{\rm out}$, *the integral does not yield elementary functions. This is not a problem per se but it does limit intuition (Wolfram Alpha gives me exponential integrals Ei for which I don't have an intuitive feel).*

Faced with these technical challenges, and recognizing that despite our mathematical difficulties planets really do have photospheres (!), we proceed with order-of-magnitude reasoning in a bid to acquire some physical intuition.

First get a sense of how much optical depth is contributed by a single scale height $H(r)$*:*

$$\tau_H(r) \sim \rho(r)\kappa H(r)\,. \tag{7}$$

This estimate is good so long as $H \ll r$*—see part d.* **As** r **decreases, does** $\tau_H(r)$ **as given by (7) increase or decrease? Justify your answer—one way to do this is to plot** $\tau_H(r)$ **vs.** r**.** *The point is to get a sense of whether the bottom scale heights or the top scale heights matter more for optical depth.*

Argue from the above that most of the optical depth above the photosphere is provided by the FIRST scale height RIGHT ABOVE the photosphere, i.e.,

$$\tau_H(r_{\rm photo}) \sim \rho\kappa H|_{\rm r_{photo}} \sim 1 \tag{8}$$

where all quantities are evaluated at the photospheric radius $r_{\rm photo}$. *This is an equation for* $r_{\rm photo}$.

Rather than solve for $r_{\rm photo}$ *directly (we'll do that later in part f),* **show that (8) can be re-written as**

$$P_{\rm photo} \sim g/\kappa \tag{9}$$

where $P_{\rm photo}$ is the photospheric pressure and g is the local gravitational acceleration. This is a handy and often-used relation (but caveat emptor part g).

(f) [5 points for all questions] Although we don't yet know exactly where $r_{\rm photo}$ is, we may make an order-of-magnitude approximation for $g = GM/r^2$ using $r \sim 3R_\oplus$. Assume further that $\kappa \sim 1$ cm^2/g (a very rough estimate assuming species that absorb strongly at optical wavelengths like atomic Fe, TiO, VO, and alkali metals Na and K). Then **estimate $P_{\rm photo}$ in units of bars**.

At what radius R does the pressure equal $P_{\rm photo}$? *This is planet's "total" radius. You will need parts b and c to calculate this.*

Finally, **comment on how sensitive your answer is to the photospheric opacity κ. What if we over-estimated κ by two orders of magnitude (so consider instead $\kappa = 0.01$ cm^2/g)? By how much would your answer for R change (if you want, you can express as a percentage)?**

(g) [5 points] Parts e and f are actually not quite appropriate for transit observations. In a transit observation, we aren't sensitive to the traditionally defined photospheric $\tau = 1$ surface measured radially to the center of the planet. We are sensitive instead to where the optical depth ALONG THE LINE OF SIGHT — call this $\tau_{\rm los}$ — reaches unity. See Figure 1. The transit radius of the planet $R_{\rm transit}$ is the radius where $\tau_{\rm los} \sim 1$. This is the radius that is actually reported in the literature on transit observations.

From Figure 1, we see that the relevant path length for $\tau_{\rm los}$ is not H (as we assumed in part e), but rather a longer chord of length $L_{\rm chord}$. **Calculate how much longer $L_{\rm chord}$ is compared to H. Express your answer for $L_{\rm chord}/H$ in terms of H and $R_{\rm transit}$. Use $H \ll R_{\rm transit}$ to simplify your answer**.

Make also a numerical estimate for $L_{\rm chord}/H$, *approximating $R_{\rm transit} \sim R$ using your answer in part f, and evaluating H at $r \sim R$.*

From these considerations **decide whether your estimate for $P_{\rm photo}$ in part f underestimates or overestimates the true pressure at the transit radius. Finally estimate by how much your answer for R in part f is in error—state whether the true $R_{\rm transit}$ is larger or smaller than your answer in part f, and express your estimate of the error as a percentage** *[e.g., 1%, 10%, 100% (i.e., a factor of 2 error), 1000% (a factor of 10 error), etc.].*

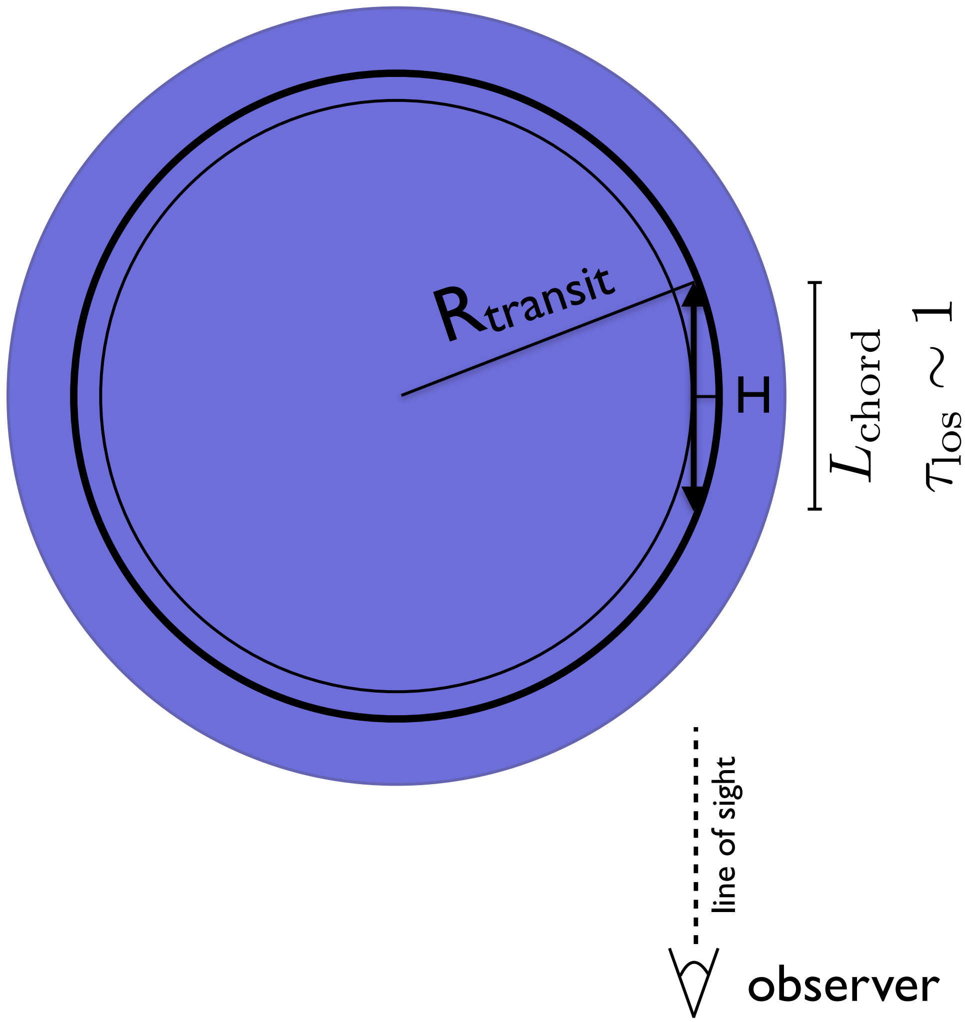

Figure 1: Transit radius. The observer is imagined to be at the bottom of the page, looking upward. The planet is a diffuse (fuzzy) object whose density drops sharply with increasing radius. At a certain $R_{\rm transit}$, the line-of-sight optical depth $\tau_{\rm los}$ presented by a single shell of radial thickness H equals 1. This marks approximately the effective transit radius of the planet. We can imagine drawing the same diagram for a shell at $r > R_{\rm transit}$; then the chord length is longer, but the density ρ falls more dramatically, so $\tau_{\rm los} < 1$. Same reasoning applies but with opposite signs for shells drawn at $r < R_{\rm transit}$.