Planetary Astrophysics – Problem Set 5

Due Thursday Oct 8

1 Clouds, Part I (Purely Absorbing, One-Dimensional)

Consider a plane-parallel atmosphere (of radial thickness $\ll$ the planet radius) near the surface of a planet. Light from the host star strikes the atmosphere at normal incidence (perpendicular to the atmospheric layers).

The atmosphere has a thick cloud layer. Clouds are composed of droplets, typically microns or fractions of a micron in size, of a condensed trace species. On Earth, the principal trace condensible species is water; on Venus, it is sulfuric acid; on hot exoplanets, it might be silicates or even irons (people talk about "forsterite clouds").

Take the cloud layer to have a constant droplet density [droplets per volume] η, the individual droplet radius to be R, and the vertical thickness of the cloud layer to be z_{max}. Measure vertical distance through the cloud by z, where the top of the cloud is located at $z = 0$ and the base of the cloud is located at $z = z_{max}$ (so z increases as you go deeper into the atmosphere).

(a) [3 points] Write down the total vertical optical depth of the cloud, τ.

(b) [4 points] Consider a thin vertical slice of the cloud Δz, over which the optical depth $\Delta\tau \ll 1$. Show that $\Delta\tau$ can be interpreted as a "covering fraction", i.e., if one were to lay this thin slice flat, the fraction of the area covered by cloud particles would be $\Delta\tau$.

(c) [3 points] Suppose the cloud particles are purely absorbing. Every stellar photon that hits a cloud particle is absorbed and vanishes forever.

When stellar photons of flux F (number per area per time) traverse the thin slice of cloud of optical depth $\Delta\tau \ll 1$, a fraction of them are absorbed away:

$$\Delta F = -F\Delta\tau \,. \tag{1}$$

This should make sense given the geometric interpretation of $\Delta\tau \ll 1$ as a covering fraction (part b). For example, if the cloud particles cover 1% of the area of the slab, then 1% of the photons passing through the slab are absorbed.

Take the differential limit of (1):

$$dF = -F\,d\tau \,. \tag{2}$$

Integrate this equation to find the (unabsorbed) stellar flux F_t that is transmitted through the cloud after traversing the full optical depth τ. Express your answer for F_t in terms of τ and the incident flux F_i at the top of the cloud.

2 Clouds, Part II (Purely Scattering, One-Dimensional)

We return to the same cloud of Problem 1, except that now we assume the cloud particles are purely SCATTERING. When a stellar photon collides with a scattering cloud particle, it is not absorbed, but just re-directed into a random direction. Photons get bounced like pinballs from cloud droplet to cloud droplet, preserving their wavelength and never getting absorbed. For this plane-parallel atmosphere where we consider only photons at normal incidence, there is equal probability that the photon will be re-directed up as down. Some photons/pinballs will be lucky enough to make it through the cloud, while some will get pinballed back into space. This problem, just like the previous problem, is interested in calculating the fraction that make it through (F_t/F_i, the ratio of transmitted to incident fluxes).

As before, take the cloud to have a constant droplet density [droplets per volume] η, the individual droplet radius to be R, and the vertical thickness of the cloud to be z_{max}. Measure vertical distance through the cloud by z, where the top of the cloud is located at $z = 0$ and the base of the cloud is located at $z = z_{max}$ (so z increases as you go deeper into the atmosphere).

(a) [0 points] Write down the vertical optical depth of the cloud, τ. This is identical to Problem 1a above. No extra points will be awarded, but you need this answer to solve later parts of this problem.

(b) [3 points] Incident photons from the host star strike the top of the cloud. The incident photons have a number flux F_i [number per time per area]. What is the number density of incident photons at the top of the cloud? Call this photon number density n_i and express in terms of F_i and fundamental constants.

These incident photons have NOT been scattered yet by any droplet. Remember the fisherman's mantra that a "flux is a number density multiplied by a speed."

(c) [3 points] These photons pinball/random walk through the scattering droplets. Random walks are described by the diffusion equation,

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial z^2} \tag{3}$$

where D is the diffusion coefficient and n is the photon number density.

Express D in terms of any of the symbols defined above and fundamental constants. Just an order-of-magnitude expression suffices.

(d) [3 points] In steady-state, $\partial n/\partial t = 0$ (the number density of photons everywhere in the cloud does not change with time). Write down the solution to the diffusion equation for $n(z)$. You should have two, as yet unknown, constants of integration. Call them A and B.

(e) [9 points] The NET number flux, F, of photons at $z = 0$ (top of the cloud) equals the incident flux F_i (directed down into the cloud) MINUS the outgoing reflected flux F_r (directed up, away from the cloud into space). We have:

$$F(z = 0) = F_i - F_r = -D\partial n/\partial z \tag{4}$$

where for the last equality we have used Fick's law, which is just another way of writing the diffusion equation.

The entire system is in steady state (does not vary in time), which imposes a condition on $F(z)$—one we discussed in class, and which you should verify is satisfied by your answer in (d).

The transmitted flux F_t is the flux at the base of the cloud, at $z = z_{\max}$, directed downward. By energy conservation, we must have that the incident flux F_i equals the reflected flux F_r PLUS the transmitted flux F_t. In other words,

$$F_i = F_r + F_t \tag{5}$$

Use all of the above, including parts (a)–(c), to calculate the transmission factor F_t/F_i, the ratio of the transmitted flux to the incident flux, in terms of τ.

Hint: begin by solving for the undetermined constants A and B in terms of F_t and F_i.

(f) [2 points] Evaluate F_t/F_i for the clouds of sulfuric acid on Venus. Use parameters appropriate for the "middle cloud" region, for which $\eta \sim 10^2\,\mathrm{cm}^{-3}$, $R \sim 3\,\mu\mathrm{m}$, and $z_{\max} \sim 5\,\mathrm{km}$ (see pages 108–109 of the Course Reader).

3 Greenhouse Warming (a.k.a. Radiative Atmospheres)

We return again to the plane-parallel atmosphere of previous problems, this time considering the planet's own thermal radiation. We consider a planet with a hard surface, like Venus or Earth.

The transmitted stellar flux F_t makes it all the way to the surface. Assume that all of it is absorbed by the ground. To maintain equilibrium, this absorbed flux must be re-radiated back into space. The re-emitted radiation is no longer at stellar optical wavelengths, but is instead reprocessed to infrared wavelengths by the ground. This ground radiation tries to escape to space. It will have a hard time getting out if the atmosphere is optically thick to infrared radiation, which it is for Venus and Earth because of "greenhouse gases" that absorb (and re-emit) strongly in the infrared. Nevertheless the re-radiation does eventually make it out — it has to, if it is to maintain radiative equilibrium. This problem calculates the temperature profile of such a "radiative atmosphere".

The atmosphere's "job" is to carry away to space a flux F ($= F_t$). Define the "effective temperature" T_{eff} such that

$$\sigma T_{\mathrm{eff}}^4 \equiv F \tag{6}$$

where σ is the Stefan-Boltzmann constant. It is crucial to remember that T_{eff} does not refer to any temperature in the atmosphere per se. It is just a convenient proxy for the flux F that must be carried away.

From class, the equation of radiative diffusion in a plane-parallel atmosphere reads:

$$F(\tau) = \frac{16\sigma T^3}{3}\frac{dT}{d\tau} \tag{7}$$

where F is the outgoing flux, and the optical depth τ increases from 0 at the top of the atmosphere (in space) to $\tau_{\max}$ at the bottom of the atmosphere. Here $T(\tau)$ really is the actual temperature of the atmosphere at optical depth τ, assuming the atmospheric gas is in local thermodynamic equilibrium with the radiation field. We emphasize that the relevant optical depth τ in this context is of the infrared-absorbing species in the atmosphere: generally this is NOT CLOUDS, but rather some gas that absorbs and re-emits strongly at infrared wavelengths. For Venus it is principally gaseous CO_2; for Earth, it is CO_2, H_2O, and CH_4.

(a) [2 points] Explain why F must be constant with τ in steady state.

(b) [3 points] Use the constancy of $F(\tau) = F = \sigma T_{\rm eff}^4$ to solve (7) for T as a function $T_{\rm eff}$, τ, and T_0 = the temperature of the atmosphere at $\tau = 0$.

(c) [2 points] To solve for T_0, consider a vertically thin slice (slab) of atmosphere near its top. This slab has optical depth $\Delta\tau \ll 1$. The outgoing flux $F = \sigma T_{\rm eff}^4$ enters the slab from the bottom. Consider just the (few) absorbers within the slab. Idealize these absorbers as perfectly flat blackbodies of temperature T_0.[1] Calculate T_0 in terms of $T_{\rm eff}$, using the fact that the assumed flat absorbers absorb from only their bottom faces but re-emit from both their top and bottom faces.

Use your answer for T_0 to re-write T from part (b) in terms of τ and $T_{\rm eff}$ only.

(d) [3 points] The effective temperature of a planet is sometimes calculated using

$$\frac{L_\star}{4\pi a^2}\pi R^2(1-A) = \sigma T_{\rm eff}^4 4\pi R^2 \tag{8}$$

which says the power absorbed by the planet (which presents an absorbing cross section $\pi R^2(1-A)$ to incident light from a star of luminosity $L_\star$ located a distance a away, where A accounts for cloud cover) equals the power it radiates away (over its entire surface area $4\pi R^2$ — this presumes that the absorbed power is efficiently re-distributed from the dayside to the nightside. In real life this re-distribution is made possible by winds.)

Calculate $T_{\rm eff}$ using (8) and values appropriate for the Earth. Use $A = 0.5$ to account for the average cloud cover. Does $T_{\rm eff}$ fit with your everyday experience of Earth's actual air temperature? Full credit for 2 significant figures.

Using this same value of $T_{\rm eff}$, calculate $T(\tau_{\max})$, using $\tau_{\max} \simeq 2$ for the Earth due to CO_2, H_2O, and CH_4. Does $T(\tau_{\max})$ seem more or less realistic than $T_{\rm eff}$?

Repeat the calculation of $T_{\rm eff}$ and $T(\tau_{\max})$ for Venus, for which $a = 0.7$ au and $\tau_{\max} \simeq 200$ (100× larger than for the Earth). Also include the fact that clouds on Venus imply $A \simeq 0.9$ (hint: compare with Problem 2f).

[1] I picture flat poker chips, oriented parallel to the slab. Because $\Delta\tau \ll 1$, the chips can be safely assumed to be non-overlapping.