Planetary Astrophysics – Problem Set 6

Due Thursday Oct 15

1 Air Conditioners, Airplanes, and Atmospheric Stability

The air outside an airplane is at a pressure $p \approx 264$ *millibars and temperature* $T \approx 223$ *K.*

(a) [5 points] Would the air outside the airplane, if adiabatically compressed, be suitable for use in a plane's passenger cabin? Take the ratio of specific heats of air to be $\gamma = c_p/c_v = 7/5$*, appropriate for classically excited, rigid diatomic rotating molecules.*

(b) [5 points] Based on your answer in (a), would you conclude that the Earth's atmosphere below the airplane is stable or unstable to convection? In other words, is the actual temperature gradient in the lower atmosphere sub-adiabatic or super-adiabatic?

2 Dry vs. Wet Adiabatic Lapse Rates

Here we derive and numerically evaluate the adiabatic temperature gradients for dry and moist atmospheres. Remember that these adiabatic gradients DO NOT EQUAL the actual gradients. The adiabatic gradients are reference quantities for any planetary atmosphere, to be compared against the actual gradient to decide whether an atmosphere is convective or not.

(a) [5 points] Use the adiabatic relation for (a dry, non-phase-changing) gas

$$P \propto \rho^{\gamma}\,, \tag{1}$$

the condition of hydrostatic equilibrium,

$$\frac{1}{\rho}\frac{dP}{dz} = -g \tag{2}$$

and the ideal gas law,

$$P = \frac{\rho k T}{\mu m_{\rm H}} \tag{3}$$

to solve for the so-called "dry adiabatic lapse rate," dT/dz*, in terms of* g*,* k*,* $\mu m_{\rm H}$*, and* γ*.*

(b) [5 points] Evaluate the dry lapse rate for conditions in the Earth's troposhere (the layer closest to the ground). Express in [K / km], and make sure you get the sign right. Assume the N_2 and O_2 in the Earth's atmosphere is rotationally but not vibrationally excited. Full credit for an answer within 25% of ours.

(c) [5 points] If, as the temperature decreases with increasing height above the atmosphere, some TRACE (minor) species (e.g., water) of the atmosphere begins to condense out and make clouds, latent heat is released by the condensing vapor. This latent heat increases the temperature of the atmosphere. Thus, the so-called "wet adiabatic lapse rate" is smaller in magnitude than the "dry" rate; the temperature of the moist atmosphere still decreases with height, but less steeply than for a dry atmosphere because of the latent heat released by condensing vapor.

The first law of thermodynamics can accommodate this extra release of latent heat:

$$dU = -P\,dV + dQ + L_{vap}\,dm \tag{4}$$

where L_{vap} [erg/gram] is the latent heat of vaporization (read: condensation, which is just the inverse of vaporization) of the trace constituent, and dm is the change in the mass of the trace species that condenses out ($dm > 0$ means more mass has condensed out). Here U, P, and V are the internal energy, pressure, and volume of our proverbial test parcel (a.k.a. blob) of gas. It is crucial to appreciate that even though $dm \neq 0$ because of condensation, the total mass of the test parcel may be approximated as constant, because the species that condenses out is assumed to be a TRACE species that adds negligibly to the total mass.

By definition, for the adiabatic processes assumed here, $dQ = 0$ (the parcel exchanges no heat with its environment).

We also know from lecture that, for an ideal gas:

- *$dU = C_V dT$ (practically by definition of C_V, the heat capacity at constant volume)*
- *$C_P = C_V + Nk$, where C_P is the heat capacity at constant pressure, and N is the total number of particles in the parcel (neglecting the trace condensible)*
- *$\gamma \equiv C_P/C_V$*

Here C_V and C_P are extensive quantities, meaning they refer to the heat capacities of the entire blob of gas; they are not measured per particle or per mass (deriving the latter specific heats from the total heat capacity is easy; just divide by N or $N\mu m_{\rm H}$). Other extensive quantities are U and V.

Note that our introduction of γ in this context DOES NOT mean that $P \propto \rho^\gamma$. That relation was true for a dry gas ($dm = 0$) that behaves adibatically. Although we are still dealing with a gas that behaves adiabatically, $dm \neq 0$ now and therefore $P \propto \rho^\gamma$ no longer holds! So don't use this relation for the remainder of this problem.

Use the 3 bullet-item ideal gas relations stated above to express dU in terms N, dT, k, and γ.

(d) [5 points] Define w to be the mass in the trace vapor that condenses out, divided by the total mass of the blob. In other words, w is the mass of condensed vapor per unit mass of total atmospheric gas; w is dimensionless.

Use your answer in part (c) plus the first law (4) to derive an expression for $d\rho/dT$ in terms of ρ, P, γ, $L_{\rm vap}$, $\mu m_{\rm H}$, and dw/dT. Here ρ is the usual mass density.

(e) [5 points] Combine the ideal gas law and the equation of hydrostatic equilibrium to derive an expression for dT/dz in terms $d\rho/dT$. Then insert your answer from (d) to derive the wet adiabatic lapse rate:

$$\frac{dT}{dz} = \frac{-g}{\frac{k}{\mu m_{\rm H}}\frac{\gamma}{\gamma-1} - L_{vap}\frac{dw}{dT}} \tag{5}$$

Remember that we are treating the condensible vapor as a trace (minor) constituent of the total atmosphere. This is a good approximation for many situations: for Jupiter, the condensible vapors are ammonia, water, and ammonium hydrosulfide (vs. molecular hydrogen and helium for the bulk of the atmosphere); for the Earth, the condensible is water (vs. nitrogen and oxygen); for Venus, the condensible is hydrosulfuric acid (vs. nitrogen and carbon dioxide). The quantity dw/dT measures the amount of vapor mass that condenses per unit mass of atmospheric gas, per degree Kelvin change. If $dT < 0$, then $dw > 0$; hence $dw/dT < 0$.

(f) [5 points] Assume that the air is completely saturated with water vapor at every height. Use the Clausius-Clapeyron formula for the saturation vapor pressure of water,

$$P_{\rm sat,water} = P_0 \exp[-T_0/T] \tag{6}$$

where $P_0 = 3 \times 10^7$ bar and $T_0 = 6144.5$ K. This gives the partial pressure of gaseous water vapor, in equilibrium with condensed liquid droplets. As T decreases, $P_{\rm sat,water}$ decreases; in other words, as it gets colder, the air can't hold as much water vapor; the water vapor is forced to condense out as droplets (= fog when the condensation happens near the ground).

By definition, the change in w is:

$$dw = -d(\rho_{\rm sat,water}/\rho_{\rm air}) \tag{7}$$

where we have inserted a minus sign because dw measures the mass in water that condenses out, whereas $\rho_{\rm sat,water}$ measures the mass in water that is still in the vapor phase (one is the minus of the other).

When evaluating dw, take the background tropospheric air density $\rho_{\rm air}$ to be constant. Since $P_{\rm sat,water}$ decreases exponentially fast with decreasing temperature in the troposphere, all the water will condense out over a short interval in height, shorter than an atmospheric scale height. Over this short interval, the background density $\rho_{\rm air}$ will be constant to good approximation.

Use all of the above, together with an average tropospheric air temperature of $T = 290$ K, to estimate dw/dT. An answer within a factor of 2 of ours gets full credit. From *there, combine with $L_{\text{vap}} = 2 \times 10^{10}$ erg/g to estimate the wet adiabatic lapse rate dT/dz. Express dT/dz in [K/km]. An answer within 50% of ours gets full credit.*

Check that you have the right magnitude for the wet lapse rate relative to the dry lapse rate. One should be larger than the other.

(g) [5 points] Do you expect moist regions in the atmosphere where vapor is condensing to be more likely to be convective than dry regions in the atmosphere? Which environment do hang-gliders prefer? In the more vigorously convective environment, what is the extra source of energy?