Planetary Astrophysics – Problem Set 7

Due Thursday October 22

1 Just How Tiny is the Tiny Superadiabatic Temperature Gradient?

Convection is so efficient at transporting heat that the actual temperature gradient in a convective atmosphere is only slightly greater than the adiabatic temperature gradient. Here we estimate quantitatively what "slightly greater" means.

Recall that the Brunt-Vaisala (B-V) frequency is the frequency of buoyant vertical motions in an atmosphere, and is given by

$$\omega_{B-V}^2 = \left[\left.\frac{\partial T}{\partial z}\right|_{actual} - \left.\frac{\partial T}{\partial z}\right|_{adiabatic} \right] \frac{g}{T}\,, \tag{1}$$

where $g > 0$ is the gravitational acceleration, T is temperature, and z measures height (increasing as one travels away from the center of the planet). Define

$$\Delta\nabla T = \left[\left.\frac{\partial T}{\partial z}\right|_{actual} - \left.\frac{\partial T}{\partial z}\right|_{adiabatic} \right] \tag{2}$$

to be the difference between the actual temperature gradient and the adiabatic temperature gradient. We will estimate $\Delta\nabla T$, and compare it to $\nabla T|_{actual}$. Remember that if $\Delta\nabla T < 0$, then $\omega_{B-V}^2 < 0$—in other words, the B-V frequency has an imaginary component, any vertical motions are unstable, and convection ensues. Since $\nabla T < 0$, $\Delta\nabla T < 0$ means the absolute value of the actual temperature gradient, $|\nabla T|_{actual}$, exceeds the absolute value of the adiabatic temperature gradient, $|\nabla T|_{adiabatic}$; we say the actual temperature gradient is **superadiabatic** *(but not by much) in convective atmospheres.*

(a) [5 points] Consider a parcel of gas moving adiabatically upwards through a convective (superadiabatic) atmosphere. The parcel has mass density ρ and specific heat c [erg/(gram K)].[1] The parcel maintains pressure equilibrium with its surroundings: as it rises, the parcel's pressure matches exactly the surrounding atmospheric pressure (which is decreasing with increasing height). The parcel's temperature decreases adiabatically,

[1] For this problem, we are deliberately not specifying whether the specific heat c is to be evaluated at constant volume or constant pressure, as the problem is intended to be done to order of magnitude, and the difference between c_v and c_p is less than a factor of 2. If we were interested in exact answers, we would need to worry about other order-unity factors, not just the difference between c_v and c_p.

while the atmosphere's temperature drops superadiabatically. In other words, the adiabatic drop in the parcel's temperature is not as much as the drop in the surrounding environment's temperature, because the actual temperature gradient of the environment is superadiabatic.

After the parcel has risen length ℓ, where ℓ is small compared to the length scale over which background quantities change,[2] *roughly how much excess thermal energy density [erg/cm^3] does the parcel carry relative to its surroundings? Use the variables given above. Call this extra thermal energy density ϵ and derive an order-of-magnitude expression for it using the variables given above.*

(b) [5 points] Give an approximate symbolic expression for the upward velocity, v, of the buoyant parcel after it has travelled distance ℓ. Remember that the parcel is unstably buoyant; it experiences an upward acceleration, $-(\delta\rho/\rho)g$, where $g > 0$ is the local (downward) planetary gravitational acceleration. You should first understand why $\delta\rho$, the density difference between the parcel and its surroundings, is negative. Reduce your expression to one that DOES NOT contain ρ or $\delta\rho$, but DOES contain T.

(c) [5 points] Assume that convection dominates heat transport through the atmosphere. The atmosphere has a job to do: it must transport an energy flux F [erg cm^{-2} s^{-1}]. We have $F \sim \epsilon v$: in general, according to the fisherman's mantra, the density of any quantity (in this case energy density ϵ) times the speed with which that quantity moves (v) gives a flux (F).

Use $F \sim \epsilon v$ and (a) and (b) to solve for an approximate symbolic expression for $\Delta\nabla T$. Your answer should depend on F, ℓ, T, g, c, and ρ.

(d) [10 points] Estimate $|(\Delta\nabla T)/(\nabla T)_{actual}|$ for conditions appropriate to Jupiter's atmosphere at a pressure of 1 bar. You may draw real numbers from the plot on page 87 of the Course Reader (taken from Chamberlain & Hunten's excellent textbook, "Theory of Planetary Atmospheres").

For this numerical evaluation, the only quantity which is not given by data is ℓ, the distance a parcel travels before it dissolves away and releases its excess energy to its surroundings. The process by which the parcel (think hot blob)[3] *decays into the surroundings is not understood in detail; convection is a form of turbulence, and we do not have a theory for turbulence. We make do with so-called "mixing length theory": we assume that $\ell \sim H$, the local scale height of the atmosphere. For H, see PS 4—in particular the heuristic derivation,*[4] *which expresses H in terms of g, T, and μm_{H} ($\mu \sim 2$ is the mean molecular weight and m_{H} is the mass of the hydrogen atom).*

To estimate F, use the flux incident on Jupiter from the Sun.[5]

[2] Later in part d we will violate this assumption—but only marginally—by taking ℓ to be of order the local scale height, the distance over which background quantities like pressure, density, and temperature change by a factor of order $e = 2.718$ ...

[3] It could also be a cold sinking blob; all the signs of this problem just reverse.

[4] In that problem set we derived H in two ways: heuristically but also physically correctly, using hydrostatic equilibrium for a collisional gas (a.k.a. fluid).

[5] On the one hand about 1/2 of the incident Solar flux is reflected away by clouds, but on the other Jupiter produces 2× as much energy as it absorbs from the Sun, so these factors tend to cancel out.

Approximate the specific heat $c \sim 2k/(\mu m_{\rm H})$, where k is Boltzmann's constant.

Is your computed dimensionless quantity $|(\Delta\nabla T)/(\nabla T)_{actual}|$ tiny?

(e) [10 points for all questions in this part] Using the above results, express ϵ in terms of ρ and v only. You can actually derive this result just using dimensional analysis, but full credit will be given only for a complete (order-of-magnitude) derivation.

What this result shows is that the excess thermal energy of the convective parcel is of order its bulk kinetic energy; thermal energy and bulk kinetic energy are in rough equipartition in convective turbulence.

Insert your new expression for ϵ in $F \sim \epsilon v$ to estimate the convective velocities v at 1 bar in Jupiter's atmosphere. Express in cm/s, and also as a fraction of the thermal speed (a.k.a. sound speed) $c_{\rm s} = \sqrt{kT/(\mu m_{\rm H})}$. As long as fluid velocities are much less than the thermal speed, the errors introduced by assuming hydrostatic equilibrium (which for a convective atmosphere is not strictly correct) are small (take a fluid dynamics class like Astro/Physics C202 to understand why).

2 Cooling Curves

Burrows & Liebert (1993) describe how a brown dwarf or giant planet's cooling luminosity L scales with time t, mass M, and photospheric opacity $\kappa_{\rm e}$:

$$L \propto t^{-1.297} M^{2.641} \kappa_{\rm e}^{0.35} \tag{3}$$

(page 78 of the Course Reader). Here time t is the time that has elapsed since all the heat of the planet was first trapped inside it, i.e., the time since the planet formed. This problem tries to derive analoguous power-law scalings using a simple model. We won't be able to reproduce the exact scalings above because our model will be too crude, but we will get qualitatively similar results.[6]

The thermal energy E of a planet that cools passively into space by radiation is given by:

$$\frac{dE}{dt} = -L = -4\pi R^2 \sigma T_{\rm e}^4 \tag{4}$$

where R is the planet's photospheric radius and $T_{\rm e}$ is its photospheric surface temperature (subscript "e" to denote the "effective" blackbody temperature of the photospheric surface). We consider the case where R is fixed in time, while E and $T_{\rm e}$ change in time (they decrease as the planet cools, like a dying ember in a fireplace).

Re-write (4) to order-of-magnitude using $dE \sim \Delta E \sim -E$ and $dt \sim \Delta t \sim t$:

$$\frac{E}{t} \sim 4\pi R^2 \sigma T_{\rm e}^4 \tag{5}$$

Comment (not necessary to solve this problem): writing (5) actually restricts the meaning of t. The meaning of t in (5) is that it is time over which E drops by an order-unity

[6] In the grad version of this class (Astro/EPS C249), we present a more realistic and complicated model that gets closer to the Burrows & Liebert result.

factor (say a factor of 2). We call this the cooling time. Ordinarily, the cooling time t is not a constant but grows as wall-clock-time elapses; it takes progressively longer for E to drop by successive factors of 2 as time elapses, since the colder an object is, the less efficiently it tends to lose its energy. The cooling time t may or may not be similar to the time $t_{\rm elapsed}$ since formation. At early times, $t_{\rm elapsed} \ll t$—we are observing the planet before its first cooling time, at such a young age that it has not had time to cool. Later, after the first cooling time, $t \sim t_{\rm elapsed}$. Ordinarily it is not possible for $t_{\rm elapsed} \gg t$; the cooling time t generally increases with time $t_{\rm elapsed}$ in such a way that $t \sim t_{\rm elapsed}$ for all times after the first cooling time. The order-of-magnitude equality $t \sim t_{\rm elapsed}$ is used often in other astronomical contexts involving passively cooling objects (e.g., cooling white dwarfs are used to age-date galaxies).

(a) [2 points] Idealize the planet interior ($\neq$ the photosphere) as being at a uniform "core" or "central" temperature $T_{\rm c}$. Write down an order-of-magnitude expression for the total thermal energy of the planet in terms of $T_{\rm c}$, M, the mean mass per particle $\mu m_{\rm H}$, and fundamental constants. Assume that the interior contains nearly all of the mass M.

(b) [2 points] In the interior, hydrogen is ionized. It is important to appreciate that the electrons and protons behave differently in a giant planet. The ELECTRONS are degenerate while the protons are not.[7] The electrons and their degeneracy pressure support the planet against gravity. Write down how R scales with M for an object supported by free electron degeneracy pressure. Combine what you have written with (a) and insert into equation (5) to find that

$$T_{\rm c} \propto t M^{-5/3} T_{\rm e}^4 \,. \tag{6}$$

(c) [5 points] Now derive another relation between $T_{\rm c}$ and $T_{\rm e}$ by using the fact that the planet is completely convective. We know that a convective object behaves nearly adiabatically (technically super-adiabatically, but we know that the degree of super-adiabaticity is small; see the previous problem). Assume the interior (from the core all the way to the photosphere) obeys a simple adiabatic relation $P \propto \rho^\gamma$, where P is pressure and ρ is mass density.[8] Then

$$\frac{P_{\rm c}}{\rho_{\rm c}^\gamma} = \frac{P_{\rm e}}{\rho_{\rm e}^\gamma} \tag{7}$$

It is here that we recognize that it is the PROTONS which behave adiabatically, i.e., the protons still behave like an ideal gas, and so $P_{\rm c}$ represents the proton thermal ideal gas pressure.

Write down:

- *how the ideal gas pressure $P_{\rm c}$ scales with $T_{\rm c}$ and $\rho_{\rm c}$*
- *how $\rho_{\rm c}$ depends on M and R (to order of magnitude)*

[7] You can show that this is true for Jupiter. Who knows, it might be on an exam. See PS 1.

[8] This is the most severe approximation made in this problem. In C249, we relax this approximation and consider how hydrogen transitions from molecular to ionized from the surface to the core. Then we don't restrict ourselves to a simple power-law relation between P and ρ.

- *how the photospheric pressure $P_{\rm e}$ depends on surface gravity g and photospheric opacity $\kappa_{\rm e}$ (see PS 4)*
- *how g scales M and R*

and insert everything, including how R scales with M (part b), into (7) to write down how $T_{\rm c}$ scales with M, $\kappa_{\rm e}$, and $T_{\rm e}$.

(d) [6 points] Combine your answer in (c) with (6) to derive how $T_{\rm e}$ scales with t, M, and $\kappa_{\rm e}$. Substitute into (4) to find how L scales with M, t, and $\kappa_{\rm e}$. Give symbolic expressions in terms of γ. Additionally, for the L scaling, evaluate the numerical exponents assuming $\gamma = 5/3$. You may compare your numerical exponents with the Burrows & Liebert (1993) exponents given by (3).