Planetary Astrophysics – Problem Set 8

Due Thursday Nov 5

1 Little Bear's Porridge

Of the terrestrial planets Venus, Earth, and Mars, why does only Earth exhibit plate tectonics? One line of argument (it is just an argument, i.e., it is simplified and not conclusive) concerns the relative thicknesses of their hard, conductive lithospheres.

Take the internal source of energy of every rocky planet to be from internal radioactivity. Assume each planet is, on average, made of chondritic (primitive meteorite) material having solar proportions of the radioactive elements ^{40}K, ^{238}U, *and* ^{232}Th *(having half-lives measured in billions of years). Chondritic material today emits* $\varepsilon_{\rm rad} \simeq 5 \times 10^{-8}\,{\rm erg\,s^{-1}\,g^{-1}}$. *Then given the total mass of each planet, one can calculate the internal luminosity of each planet.*[1]

This internal energy is carried conductively through the hard outer lithosphere of each planet. Recall the equation for the conductive heat flux:

$$F = -k_{\rm c} \nabla T \tag{1}$$

where $k_{\rm c} \simeq 4 \times 10^5$ *[cgs] is the thermal conductivity (proportional to, but having different units from, the thermal diffusivity), and* T *is the temperature. Flux* F *has units of* ${\rm erg\,s^{-1}\,cm^{-2}}$.

At the top of the lithosphere, the temperature is just the surface temperature.

At the bottom of the lithosphere (top of the convective mantle), the temperature is ∼1200 K, the temperature at which rock starts to flow / become plastic (the viscosity of rock is

[1] What about the contribution from the heat of formation? That contribution appears a bit uncertain for the Earth. At most "secular cooling" from the heat of formation could be ∼50% of the Earth's total luminosity. It could be less. Those sources that cite ∼50% do so because the Earth's mantle appears depleted in K, which is a relatively volatile element (easier to vaporize and escape into space). Not having K means the Earth's radioactive luminosity should be only ∼50% of what one would compute by assuming the full solar complement of radioactive elements. Yet the Earth's total measured luminosity actually matches what you would compute if one assumed it had the full solar allotment. According to this line of reasoning, the extra energy that is responsible for this agreement derives from the heat of formation. However, Berkeley scientists Kanani Lee and Raymond Jeanloz have found evidence that K might not actually be depleted in the bulk Earth. They found that under high enough pressures, K may bind to Fe to become an alloy. This Fe-K alloy would sink to the core. In this picture, the Earth has the full allotment of solar K, but it is just hidden away in the core. Then pretty much all of the Earth's measured power would be accounted for by radioactivity, not from the heat of formation.

exponentially sensitive to temperature).[2]

[10 points total] Use the above, and whatever other basic data you need (e.g., planet masses and surface temperatures; you can look these up), to estimate the thicknesses $\ell_{\rm litho}$ *of the lithospheres for Venus, Earth, and Mars. Express in km. Arrange these lengths in order of increasing thickness. Answers within 50% of ours receive full credit.*

Geologists tells us that "thin" lithospheres are not good for plate tectonics because they are too light and therefore do not subduct (sink) to greater depths (like trying to drown a rubber duck; it just won't go down.) Geologists also tell us that "thick" lithospheres are not good for plate tectonics because they are, well, too thick—upwelling magma cannot break through the crust.[3]

2 Warm Feet

The mid-Atlantic ridge represents an upwelling boundary between two convection cells. New material flows horizontally away from the ridge at a roughly constant velocity. The material forms a crust across which heat is conducted (not convected) vertically.

Take the temperature difference vertically across the crust to be fixed. This temperature difference is between the top of the crust (the temperature of water at the bottom of the ocean) and the bottom of the crust (roughly 1200 K, the temperature at which rock starts to flow appreciably; same set-up was used in the previous problem).

[5 points] The vertical conductive heat flux F *varies with horizontal distance* x *from the ridge. Write down how* F *scales with* x *(a proportionality is sufficient). Remember that the crust grows in vertical thickness (*d*) as it moves away from the ridge.*

Such a scaling of F *with distance* x *is actually measured, and is one of the pieces of evidence we have that plate tectonics works the way we think it does. If you stand on the ocean floor, your feet feel hotter closer to the ridge than farther from the ridge.*[4]

3 The Four Seasons

The Earth has seasons because of its non-zero obliquity—the tilt of the planet's spin angular momentum vector (a.k.a. the spin axis) relative to the orbital angular momentum vector (a.k.a. the orbit normal). The Earth's obliquity is currently 23.5 deg.

[2] But the Earth's mantle below the lithosphere is NOT molten; it is NOT composed of liquid magma. We know the mantle is still solid because we observe it to support seismic shear waves ("S" waves; a liquid has zero shear modulus). The Earth's mantle behaves as a solid on short timescales (e.g., seismic wave periods) but like a viscous fluid on long timescales (the flow "creeps" across lengthscales spanning the entire Earth over hundreds of millions of years). Although solid, the mantle undergoes convection—"solid-state convection."

[3] Another factor that gets airtime in the discussion of whether planets exhibit plate tectonics is whether the rock is hydrated. Apparently inserting water molecules into silicate molecular lattices by just a few parts in 100 can substantially decrease the viscosity of rock and thereby promote and increase the vigor of convection. Earth's rocks are hydrated whereas Venus's are bone-dry because all of its water was lost to space in the runaway greenhouse.

[4] Not because the temperature of the ground is different (remember, the ocean floor is fixed at a single T) but because more heat is conducted through the ocean floor to your feet. It's for a similar reason that metal is cooler to the touch than wood—the metal and wood are at the same temperature (room temperature), but more heat is conducted away from your body (which is at body temperature ¿ room temperature) because metal is more heat-conductive than wood.

This problem presents a simple model for how the atmospheric temperature varies seasonally over an orbit. The model is oversimplified because it neglects horizontal heat transport, convection, and radiative transfer. But it illustrates the concept of thermal inertia, and gets answers within an order-of-magnitude of reality.

Consider a vertical column of atmosphere at fixed latitude on a rocky planet with non-zero obliquity (say the atmosphere right above Berkeley). Assume this atmosphere is vertically isothermal at temperature T, and that the atmosphere is optically thick, radiating to space (from its infrared photosphere, above the ground) with a flux σT^4, where σ is the Stefan-Boltzmann constant. σT^4 is the energy flux lost to space. In general it varies with time t because the temperature $T(t)$ is a function of time.

In addition to losing energy, the atmosphere also gains energy. It gains energy at a rate set by the incident stellar flux (at the given latitude). Call this input energy flux $F(t)$. This input flux varies as the angle of stellar insolation changes with orbital phase (e.g., F is highest at summer solstice and lowest at winter solstice).

If $F > \sigma T^4$, the atmosphere gains energy in the net, and T rises. If $F < \sigma T^4$, the atmosphere loses energy in the net, and T falls.

At any given instant in time, the energy contained in the entire column of atmosphere (units of energy/area) is given by

$$\begin{aligned} E(t) &= \int_0^\infty \rho\, c\, T(t)\, dz \\ &= T(t) \int_0^\infty \rho\, c\, dz \\ &= T(t)\, I \end{aligned} \tag{2}$$

where c is the specific heat (erg/g/K), ρ is the atmospheric mass density, and we have defined $I \equiv \int_0^\infty \rho\, c\, dz$ to be the "thermal inertia" of the atmosphere, integrated over its height z.

Combining all of the above, we have

$$\begin{aligned} \frac{dE}{dt} &= F - \sigma T^4 \\ I\frac{dT}{dt} &= F - \sigma T^4 \end{aligned} \tag{3}$$

(a) [5 points] Assume the time variations of F are small:

$$F = F_0 + \delta F \cos(\omega t) \tag{4}$$

where $\delta F \ll F_0$ are constants. The driving frequency $\omega = 2\pi/P$, where P is the planet's orbital period. We call $\delta F \cos(\omega t)$ the perturbation flux.

If the perturbation flux is small, it follows that the temperature variations throughout the year are also small:

$$T = T_0 + \delta T(t) \tag{5}$$

where T_0 is a constant, δT is not a constant, and $\delta T \ll T_0$. We call δT the perturbation temperature.

The subscript 0 denotes the average, background, time-independent ("dc") part of the atmosphere's behavior. We have:

$$F_0 = \sigma T_0^4 \tag{6}$$

since heating perfectly balances cooling in the background steady state. On top of the background state, we have the time-varying ("ac") part of the atmosphere's behavior, annotated with δ's.

Linearize equation (3) to obtain the master perturbation equation:

$$I \frac{d(\delta T)}{dt} + 4\sigma T_0^3 \delta T = \delta F \cos(\omega t) \tag{7}$$

"Linearize" means "Taylor expand T, subtract off the background terms because they cancel each other, and keep only the perturbation terms."

(b) [10 points] Try a solution of the form $\delta T / T_0 = \Delta \cos(\omega t - \phi)$, where Δ and ϕ are constants. Prove that

$$\tan \phi = \frac{\omega I T_0}{4 F_0}\,, \tag{8}$$

and that

$$\Delta = \frac{\cos \phi}{4} \frac{\delta F}{F_0}\,. \tag{9}$$

Hint: $\sin \omega t$ and $\cos \omega t$ are orthogonal functions. That means if $A \sin \omega t + B \cos \omega t = 0$ with A and B both constants, then the only way this equation can hold for arbitrary t is for $A = 0$ and $B = 0$.

(c) [10 points] Estimate $IT_0/(4F_0)$, $\delta F/F_0$, ϕ, and Δ for the air above Berkeley (answers within a factor of 2 of ours receive full credit). Insert ϕ and Δ into your solution for the fractional seasonal temperature variation, $\delta T/T_0$, and comment on how your results compare to reality.

Approximations and data you may use:

- *Berkeley's latitude is 37° N.*
- *When evaluating I, take the density of air to be constant over the first atmospheric scale height h, and neglect heights $z > h$ (the errors introduced by these two approximations tend to cancel; taking the density to be constant with z overestimates I, but omitting $z > h$ underestimates I).*
- *The density of air at ground level is about $\rho \sim 10^{-3}$ g/cm^3.*
- *The specific heat $c \sim 2k/(\mu m_{\rm H})$.*
- *For F_0, consider a surface that absorbs 1/3 of the incident solar flux, at high noon of either spring or fall equinox, at Berkeley's latitude. (The 1/3 tries to account for how the ground and clouds reflect solar radiation back into space.)*

(d) [10 points for all questions] Consider an atmosphere in steady state, with input and output energy fluxes in balance, $F_0 = \sigma T_0^4$. Now perturb the temperature of the atmosphere (everywhere) by $\delta T \ll T_0$. The atmosphere now has a perturbation energy content of δE. But it also now has a perturbation (out of balance) radiative output flux of δO, which tries to radiate away (get rid of) the perturbation energy. The "O" is meant to remind you that this is the perturbation OUTPUT flux. (I could have called it δF, but that would confuse it with the perturbation driving flux considered in previous parts of this problem.)

Derive an expression for the timescale over which the perturbation energy is eliminated by the perturbation output flux:

$$t_{\rm cool} = \delta E/\delta O$$

keeping only terms to first order in δT when evaluating δE and δO *(this is called linear perturbation theory). Express $t_{\rm cool}$ in terms of I, T_0 and F_0 and KEEP ALL ORDER-UNITY CONSTANTS. $t_{\rm cool}$ should be conceptually familiar to you as a kind of cooling time—in this case, the time to cool away the perturbation.*

Does your expression for $t_{\rm cool}$ look like anything in previous parts of this problem?

Consider the asymptotic limit that the thermal inertia I—equivalently the atmospheric density ρ—increases without bound. What does $t_{\rm cool}$ do? And what happens to ϕ and Δ in this limit? Venus has an atmosphere roughly 100 × denser than Earth, and extrasolar sub-Neptunes have atmospheres still denser by orders of magnitude. By this reasoning, are Venusian and sub-Neptune seasonal temperature variations more or less dramatic than Earth seasonal variations?