Planetary Astrophysics – Problem Set 9

Due Thursday Nov 12

1 Live from the Ancient Solar System: Aluminum-26

Lee et al. (1976, Geophys. Rev. Letters) analyzed Aluminum (Al) and Magnesium (Mg) abundances in a calcium-aluminum inclusion (CAI) in the Allende chondritic meteorite. Different minerals in the CAI (fassaite, melilite, anorthite-B, anorthite-G) give different isotope abundance ratios. But together these ratios form a trend: higher abundances of ^{26}*Mg correlate with higher abundances of* ^{27}*Al. The data look something like this:*

MINERAL	$^{27}Al/^{24}Mg$	$^{26}Mg/^{24}Mg$
Fassaite	*0*	*0.1400*
Melilite	*10*	*0.1405*
Anorthite-B	*133*	*0.1467*
Anorthite-G	*240*	*0.1520*

Here $^{27}Al/^{24}Mg$ *is the number abundance of* ^{27}*Al atoms relative to the number abundance of* ^{24}*Mg atoms. The same notation applies to* $^{26}Mg/^{24}Mg$*. You can verify that these 4 points make a pretty good line.*

It is known that ^{26}*Al is an unstable isotope of Aluminum that beta-decays with a half-life of just* 0.7×10^6 yr *into* ^{26}*Mg. Thus, some of the* ^{26}*Mg found in the CAI may be radiogenic (derived from* ^{26}*Al). By contrast,* ^{24}*Mg is non-radiogenic and is used here as a reference. Furthermore,* ^{27}*Al is completely stable.*

[10 points] Estimate the $^{26}Al/^{27}Al$ *abundance ratio at the time the CAI solidified.*[1]

2 Melting Vesta

Part (b) of this problem requires the solution to the previous Problem 1.

Vesta is an asteroid of mass $M = 2.6 \times 10^{23}$ *g and radius* $R = 260$ *km whose interior may have "differentiated," i.e., had its elements separate out with depth, with denser species sinking deeper. Differentiation is inferred from the light basaltic crust (*$\neq$ *heavier*

[1] All of the live Al-26 at the time the CAI solidified (4.56 Bya ≡ 4.56 billion years ago) is thought to have been sprayed into our solar system during its formation by a nearby supernova.

iron or iron-rich silicates) that we know to be present on the surface of Vesta. To have differentiated in this way, practically all of Vesta must have been hot enough to melt.

For rock to melt, first its temperature needs to be raised to the liquidus of ∼1200 K (melting threshold temperature). The specific heat of rock is $c \approx 1.5 \times 10^7$ *erg/g/K.*[2] *Second, enough heat must be added to convert solid to liquid (at fixed temperature; the heat does not go into raising the temperature but rather into breaking bonds). The latent heat of fusion of rock is* $L_{\rm f} \approx 4 \times 10^9$ *erg/g.*[3]

Where does the heat source required to melt Vesta come from?

(a) [5 points] Show using an order-of-magnitude argument that the heat of formation (a.k.a. heat of assembly or the heat of accretion) is inadequate to melt all of Vesta. Take Vesta's starting temperature to be that of a blackbody heated by the Sun at a distance of 2 AU.

(b) [5 points] Assume Vesta was assembled early in the solar system's history, so early that it contained the full complement of live radioactive Aluminum-26 inferred to be present from primitive meteorites (and presumably created by a nearby core-collapse supernova in the parent molecular cloud from which the Solar System formed). "Full complement" means an abundance ratio of ^{26}*Al relative to* ^{27}*Al of X by number (*^{27}*Al is the common, non-radioactive isotope of Al). We solved for X in Problem 1 of this problem set. The radioactive decay of each* ^{26}*Al nucleus into* ^{26}*Mg releases a gamma-ray photon having 1.8 MeV (mega-electron-volts). The half-life of* ^{26}*Al is* $t_{1/2} \sim 7 \times 10^5\,{\rm yr}$*, much shorter than the age of the solar system; this is a heat source that was only present at the time of formation and is no longer active.*

Assume that all of the energy released by the radioactive decay of ^{26}*Al is trapped in Vesta. Estimate the fraction of Vesta that melts due to this short-lived radionuclide. Is* ^{26}*Al promising as a heat source?*

You will need to make a reasonable model for the relevant relative elemental abundances in Vesta. You may assume that a representative mineral for Vesta, considered in bulk, is olivine ($MgFeSiO_4$*). The table of meteoritic abundances by Grevesse & Sauval (1998) on page 56 of the Course Reader may be helpful. Note that Grevesse & Sauval omit O from their list of meteoritic abundances not because O is not present in meteorites (O is plenty abundant in rocks) but because O in rocks is under-abundant relative to O in the Sun, and the focus of their paper is on solar abundances, not meteoritic abundances. In other words, meteoritic abundances are interesting to Grevesse & Sauval only insofar as they help constrain solar abundances. The underlying cosmogonic idea is that meteorites*

[2] For those who like to do order-of-magnitude problems: you can try comparing this empirical value to our order-of-magnitude formula in class, $c \sim 3k/(\mu m_{\rm H})$, and for μ take a representative metal atom like Si (atomic weight 28). The coefficient of $3 = 1/2 \times 6$, where there are 6 quadratic degrees of freedom (3 for momentum and 3 for elastic potential energy) associated with a solid, and the 1/2 comes from the equipartition theorem in thermal physics which gives $1/2 \times (kT)$ for every quadratic degree of freedom in the Hamiltonian.

[3] The order-of-magnitude rule of thumb is that the latent heat of fusion (converting solid to liquid) is 1/10 the latent heat of vaporization (converting liquid to vapor). A liquid is a solid that has enough bonds broken (about 1/10 of them) to have lost its long-range but not its short-range order, whereas a gas has lost all order. The latent heat of vaporization can be remembered as roughly ∼3 eV (atomic binding energy) / $(\mu m_{\rm H})$. Try these rules of thumb and see if they work for the empirical data given in this problem.

are made of the same "stuff" that went into making the Sun, but only some of the O in that "stuff" was able to be incorporated into rocks; the limiting ingredient in rocks is not O (which is the most cosmically abundant element by number after H and He) but a less abundant element like Fe, Mg, or Si. So use one of the latter 3 elements.

(c) [5 points] Justify to order-of-magnitude the assumption in (b) that most of the energy released by ^{26}Al is trapped in Vesta and that little of it gets radiated away into space. How long does it take the radioactive heat to conduct from the interior to the surface? Use a thermal diffusivity for rock of $D \sim 10^{-2}$ cm^2/s.[4]

3 Protostellar Disk Sizes

[10 points] The gravitational collapse of a spinning molecular cloud to form a star leads inevitably to the formation of a disk that contains all of the initial angular momentum of the cloud. In the absence of external torques—and there is no external torque acting on an isolated collapsing cloud—angular momentum is conserved.

Consider a collapsing cloud of mass $M = 1M_\odot$, initial radius $R = 0.05\,\mathrm{pc}$ (1 pc = 1 parsec = 3×10^{18} cm), and initial spin rate $\omega = 1\,\mathrm{km\,s^{-1}\,pc^{-1}}$ (clouds having these properties been observed via radio observations). As this cloud collapses inward to form a star, the outer radius of the cloud will decrease. The outer radius will not decrease indefinitely, however: it will stall at the "centrifugal barrier," when rotation (the centrifugal force) balances gravity. Estimate to order-of-magnitude the radius of the centrifugal barrier, i.e., the radius of the protostellar disk R_{disk}. Express R_{disk} in terms of the given variables and fundamental constants, and evaluate numerically in units of [AU]. Numerical answers within a factor of 10 of ours receive full credit.

4 Making Ice Cubes in the Protoplanetary Disk

Consider a "minimum-mass disk" like that discussed in class, having a mass density

$$\rho \sim 10^{-9}\,\mathrm{g/cm^3}\left(\frac{a}{\mathrm{AU}}\right)^{-39/14} \qquad (1)$$

and a temperature

$$T \sim 150\,\mathrm{K}\left(\frac{a}{\mathrm{AU}}\right)^{-3/7} \qquad (2)$$

at disk radius a. The disk is of solar composition.

(a) [5 points] At $a = 5\,\mathrm{AU}$ (the orbital radius of Jupiter), estimate the mass density ρ_{H2O} (units of mass per volume) in water vapor. Answers within a factor of 3 of ours receive full credit.

Assume that all of the cosmically abundant oxygen in the disk takes the form of water (H_2O). You may consult the Course Reader for a list of solar elemental abundances. Alternatively, you can use the rough "cosmic" proportions for rock:ice:gas = 1:3:400 (by mass), where "ice" is understood to be dominated by water.

[4] Of the same order as the thermal diffusivity of a turkey.

(b) [5 points] Estimate the mean thermal speed v_{T} of water molecules in the disk at this same stellocentric distance of 5 AU*. Give your answer in [km/s]. Answers within a factor of 3 of ours receive full credit.*

(c) [5 points] Consider the growth of an ice particle, of internal density ρ_{particle} and size (radius) r, at 5 AU. Assume that the particle accretes water molecules from the gas phase such that every gas-phase water molecule that collides with the ice particle sticks with 100% efficiency (by chemical/electrostatic forces). Assume further that the relative velocity between the ice particle and the gas-phase water molecules is dominated by the thermal motions of the latter.

Estimate the rate $\dot{r}$ at which the particle radius increases. Express $\dot{r}$ symbolically in terms of all the variables above, and numerically in units of [cm/yr]. Answers within a factor of 3 of ours receive full credit.