Problem 1. Faraday Rotation

Consider the propagation of light through a magnetized plasma. The magnetic field is uniform $\vec{B}_0 = B_0 \hat{z}$. Light travels parallel to $\hat{z}$.

An electron in the plasma feels a force from the electromagnetic wave, and a force from the externally imposed $B$-field. Its equation of motion reads

$$m \ddot{\vec{v}} = -e \vec{E} - \frac{e}{c} \vec{v} \times \vec{B}_0$$  \hspace{1cm} (1)

where the electric field $\vec{E}$ can be decomposed into right-circularly-polarized (RCP) and left-circularly-polarized (LCP) waves:

$$\vec{E} = E_0 (\hat{x} \pm i \hat{y}) e^{i(k \mp z - \omega t)}$$  \hspace{1cm} (2)

where it is understood that the real part should be taken. The upper sign (-) corresponds to RCP waves, while the lower sign (+) corresponds to LCP waves.

In the equation of motion above, we have neglected the Lorentz force from the wave’s $B$-field, since it is small (by $v/c$) compared to the force from the wave’s $E$-field.

(a) Prove that the solution of the equation of motion reads

$$\vec{v} = \frac{-ie}{m(\omega \pm \omega_{cyc})} \vec{E}$$  \hspace{1cm} (3)

where $\omega_{cyc} = eB_0/mc$. This is the hardest part of the derivation for the dispersion relation of RCP and LCP waves. But it is fairly straightforward.

The $x$-component of the equation of motion (1) reads

$$\dot{v}_x = -\frac{e E_0}{m} e^{i(k \mp z - \omega t)} - \omega_{cyc} v_y$$  \hspace{1cm} (4)

where $\omega_{cyc} = eB_0/mc$, and the $y$-component reads
\[ \dot{v}_y = \pm \frac{ieE_0}{m} e^{i(kz - \omega t)} + \omega_{\text{cyc}} \dot{v}_x \]  

These are coupled first-order ODEs. We take another time derivative of the x-equation, and then plug in the y-equation. We get

\[ \ddot{v}_x = \frac{ieE_0}{m} e^{i(kz - \omega t)} (\omega \mp \omega_{\text{cyc}}) - \omega_{\text{cyc}}^2 v_x \]  

Our ansatz (read: guess) for the solution reads

\[ v_x = Ae^{i(kz - \omega t)} \]  

which we insert into (6) to solve for A. We obtain

\[ A = \frac{ieE_0}{m(\omega_{\text{cyc}} + \omega)(\omega_{\text{cyc}} - \omega)} \]  

\[ = \frac{-ieE_0}{m(\omega \pm \omega_{\text{cyc}})} \]  

Repeat the above analysis for \( v_y \). We get

\[ v_y = Be^{i(kz - \omega t)} \]  

where

\[ B = \frac{-eE_0}{m(\omega_{\text{cyc}} \pm \omega)} \]  

It is straightforward to finish the proof from here.

(b) Rybicki & Lightman Problem 8.3

See RL’s solution on page 357.

Problem 2. Combined Scattering and Absorption

Rybicki & Lightman Problem 1.10
Problem 3. Greenhouse Warming

Taken from Chamberlain & Hunten Problem 1.3

Assume that solar radiation is absorbed only at the Earth’s surface where the albedo is 40%. The re-radiated energy is absorbed mainly by water vapor, which we approximate as a gray absorber with a density scale height of 2 km and total optical depth \( \tau = 2 \). Plot the temperature distribution with height for radiative equilibrium. What is the temperature discontinuity at the ground? What is the gradient in the air temperature near the ground, in K/km?

First calculate the effective temperature \( T_{\text{eff}} \), defined in terms of the absorbed flux:

\[
(1 - A) \times \frac{L_{\odot}}{4\pi a^2} \pi R_{\odot}^2 = \sigma T_{\text{eff}}^4 \times 4\pi R_{\odot}^2
\]

where we have assumed that the atmosphere/earth is spherically symmetric. (You could, alternatively, solve this problem just for a patch on the Earth at a certain latitude, longitude, and time, in which case you would have to calculate the local angle of insolation.) Keep in mind, the effective temperature is not an actual physical temperature; it is merely shorthand for the radiative flux.

For \( a = 1 \) AU and \( A = 0.40 \), the above equation yields \( T_{\text{eff}} = 247 \) K.

We know from lecture/reading that the temperature at infinity for an atmosphere in radiative equilibrium is \( T_0 = T_{\text{eff}}/2^{0.25} = 208 \) K.

The temperature profile according to the Eddington solution is

\[
T = T_0(1 + 3\tau/2)^{1/4} = 208(1 + 3\tau/2)^{1/4} K
\]

So at the surface where \( \tau = 2 \), the air temperature is \( T_1 = 294 \) K, which sounds about right (21 degrees Celsius).

Re: the gradient in air temperature:

\[
\frac{dT}{dz} = \frac{dT}{d\tau} \frac{d\tau}{dz} = \frac{3T}{8(1 + 3\tau/2)^{3/4}} \frac{d\tau}{dz}
\]

We are told that the density, and therefore the optical depth, falls with a scale height of \( h = 2 \) km. So \( \tau = \tau_1 \exp(-z/h) \) where \( \tau_1 = 2 \), and so \( d\tau/dz = -\tau/h \). Plugging in,
\[
\frac{dT}{dz} = -\frac{3T}{8(1 + 3\tau/2)} \frac{\tau}{h}
\]  

(15)

which at the surface gives \(dT/dz = -27.6 \text{ K/km.}\)

Re: the temperature discontinuity with the ground. From lecture/reading, we know that under conditions of M.R.E. (monochromatic radiative equilibrium) \(B_\nu(T_g) = B_\nu(T_1) + F_\nu/2\pi,\) where \(F_\nu\) is the radiative flux (constant with height). We generalize this statement for B.R.E. (bolometric radiative equilibrium), and write

\[
\sigma T_{g}^4 = \sigma T_{1}^4 + F/2 = \sigma T_{1}^4 + \sigma T_{\text{eff}}^4/2
\]  

(16)

from which it follows that \(T_g = 311\) K. So the temperature discontinuity is \(T_g - T_1 = 17\) K.