Ay 201 – Radiative Processes – Problem Set 5 Solutions
Linda Strubbe and Eugene Chiang
October 2, 2003

Problem 1. Energy Density of Starlight and Grain Temperature

Interstellar space is filled with radiation. The bulk of the radiation arises from light emitted by the most massive (O-type) stars, each of mass $10^2 M_\odot$. The number of such stars in our Galaxy is about 5×10^4, distributed over a cylindrical disc of radius 50 kpc and height 200 pc.

(a) Estimate the energy density of starlight in the Galaxy. Express your answer in eV/cm^3. To estimate the luminosity of the most massive stars, use the fact that they are radiating at the Eddington limit; i.e., recall problem set 3.

Energy density (u) is related to flux (F) by the following equation:

$$u = \frac{F}{c}, \tag{1}$$

and the flux from an object with luminosity L is

$$F = \frac{L}{4\pi r^2}, \tag{2}$$

where r is the distance to the object. Our strategy in this problem is thus to find the total flux at an "average" point in the Galaxy by adding up the flux received at that point from each O-star in the Galaxy.

To do this, we need a model to describe the locations of all the O-stars in the Galaxy. Let's approximate the Galaxy as a cylinder, with all the O-stars uniformly distributed within the Galaxy's volume. To have an idea of how dense the Galaxy is in O-stars, let's find the average distance (r_{near}) between an observer in the Galaxy and the nearest O-star. Let $N \equiv$ the number of Galactic O-stars, $r_{Gal} \equiv$ the radius of the Galaxy, $H \equiv$ the height of the Galaxy, and $V_{Gal} \equiv$ the volume of the Galaxy.

$$r_{near} \sim \left(\frac{V_{Gal}}{N}\right)^{1/3} = \left(\frac{\pi r_{Gal}^2 H}{N}\right)^{1/3} = 320\,\mathrm{pc} > H. \tag{3}$$

Since the distance to the nearest O-star is greater than the height of the Galaxy, we can make the approximation that all O-stars lie in a plane. (The radial component of the distance to *any* star is greater (and usually much greater) than the component of the distance normal to the disk.) Let's consider an observer at the center of the

Galaxy. Beyond a distance of about r_{near}, the fact that the stars are point sources is unimportant. Let's therefore model the Galaxy as a 2-dimensional annulus with inner radius of r_{near} and outer radius of r_{Gal}, which has a *continuous* uniform distribution of luminosity sources (the individual stars are smeared out over the whole area). Then we can define a luminosity surface density σ :

$$\sigma \equiv \frac{L_{O-star}N}{\pi r_{Gal}^2} \tag{4}$$

We can calculate L_{O-star} using the relationship between mass and luminosity we found in problem set 3, question 1 (p47 in Rybicki & Lightman) :

$$L_{EDD} = 1.25 \times 10^{38}\,\mathrm{erg\,s^{-1}} \left(\frac{M}{M_\odot}\right) \tag{5}$$

with $M_{O-star} = 10^2 M_\odot$, so

$$L_{O-star} \sim 10^{40}\,\mathrm{erg\,s^{-1}}. \tag{6}$$

Now we can calculate the total flux received at the center of the Galaxy:

$$dF = \frac{\sigma dA}{4\pi r^2} \tag{7}$$

$$F = \int_{r_{near}}^{r_{Gal}} \frac{\sigma}{4\pi r^2} 2\pi r\, dr = \frac{\sigma}{2} ln\left(\frac{r_{Gal}}{r_{near}}\right) \tag{8}$$

So we can finally obtain the energy density at the Galactic center:

$$u = \frac{F}{c} = \frac{1}{2}\frac{L_{O-star}N}{c\pi r_{Gal}^2} ln\left(\frac{r_{Gal}}{r_{near}}\right) \tag{9}$$

$$u = \frac{(10^{40}\,\mathrm{erg\,s^{-1}})(5\times 10^4) ln\left(\frac{50}{.32}\right)}{2(3\times 10^{10}\,\mathrm{cm\,s^{-1}})(1.5\times 10^{23}\,\mathrm{cm})^2(1.6\times 10^{-12}\,\mathrm{ergeV^{-1}})} = 0.4\mathrm{eV\,cm^{-3}} \tag{10}$$

In truth, the logarithmic correction factor over-estimates the true correction factor because stars from the other side of the Galaxy are obscured from view by dust; recall the slide show of the first class, during which I said that most of the visible starlight originates from within ∼1 kpc of Earth. Also see the next problem where we show that at visible wavelengths, we suffer about 1 mag extinction for every kpc travelled. But all logarithms are of order unity anyway, so the error accrued in including every O-star is negligible; compare the flux from the single nearest O-star and the flux from all O-stars and you will see that they differ by a factor of 5.

(b) The interstellar radiation field (ISRF) heats dust grains in the interstellar medium (ISM). Estimate the temperature, T, of the largest grains in the ISM. These grains have radii of $a \sim$ 0.1 μm. Take their emissivity (Q_{emis}) = absorptivity (Q_{abs}) to be unity for wavelengths shorter than $2\pi a$ and to fall off as $2\pi a/\lambda$ for longer wavelengths. Is the grain hotter, cooler, or equal to the temperature of an ideal blackbody placed in the ISRF?

Most of the interstellar radiation field (ISRF) is produced by O-stars, which have temperature $T \sim 40000$K. From the Wien Peak Law, we find the peak wavelength of the ISRF:

$$\lambda_{peak} = \frac{0.29\,\mathrm{cm\,K}}{T} = 0.07\mu m \tag{11}$$

In class we learned that about 95% of the intensity from a blackbody comes from the range of $\frac{1}{3}\lambda_{peak}$ to $3\lambda_{peak}$, so most $B_\lambda(ISRF)$ comes from $.02\mu m < \lambda < 2\mu m$. Thus almost all of the energy absorbed by our grain from the ISRF is in the wavelength range $\lambda < 2\pi a$ where $Q_{abs} = 1$, so the grain absorbs energy from the ISRF pretty much perfectly:

$$F_{abs} = F_{\mathrm{from\ O-stars}} = cu = 1.2 \times 10^{10} \mathrm{eV\,s^{-1}\,cm^{-2}} = 0.02\,\mathrm{erg\,s^{-1}\,cm^{-2}}. \tag{12}$$

The grain is likely to be emitting most of its power at infrared wavelengths that are much greater than $2\pi a$. Then we can approximate the emitted flux as

$$F_{emis} \approx \pi \int_0^\infty B_\lambda(T_{grain}) \frac{2\pi a}{\lambda}\, d\lambda \tag{13}$$

Here we have assumed that most of the area (power) underneath the Planck function is at $\lambda \gg 2\pi a$, so the error accrued at $\lambda \leq 2\pi a$—where our expression for Q_{emis} is incorrect, because it should be equal to 1—is negligible. Guessing different temperatures in Mathematica gives $T \approx 19\,\mathrm{K}$.

We can arrive at this result without having to resort to numerics. Write $F_{emis} = \sigma T^4 Q_{emis}$. Now $Q_{emis} = 2\pi a/\lambda$ for $\lambda > 2\pi a$. The wavelengths at which most of the power from our grain will be emitted are near $\lambda \approx hc/5kT$ from the Wien peak law. Now use of the Wien peak law is not rigorously justified because the Wien peak law is for blackbodies and our grain is not a blackbody. Still, the grain will be at some temperature T and it will be emitting most of its power at wavelengths which are inversely proportional to T; the numerical value of "5" in the Wien peak law should shift to a slightly lower number because our grain's power at long wavelengths is diluted, and therefore most of the power will get shifted to shorter wavelengths. Let's stick with "5" and see how our answer depends on this guess. Then $F_{emis} = \sigma T^4 2\pi a/(hc/5kT)$, which we set equal to F_{abs} above. This yields $T \approx 17K$, and it scales as "5"$^{-1/5}$—not that sensitively.

The temperature of an ideal blackbody placed in the ISRF is

$$F_{\mathrm{fromO-stars}} = \sigma T_{ideal}^4 \Rightarrow T_{ideal} = 4K \tag{14}$$

Our grain is thus hotter than a blackbody at thermal equilibrium in the ISRF, as it must be since it absorbs with perfect efficiency but emits with imperfect efficiency; to maintain global energy balance, the grain must boost its temperature above that of a blackbody.

(c) At what wavelength, λ_{peak}, does νF_ν (the SED; recall problem set 3) of such grains peak? Give a number in microns, and also an expression in terms of T, fundamental constants, and dimensionless numbers.

We can continue our approximation that all energy is radiated from the grain in the $Q_{emis} = \frac{2\pi a}{\lambda} \propto \nu$ regime. To find λ_{peak} of νF_ν, we find ν_{peak} by setting the derivative of νF_ν to zero, and then converting from ν_{peak} to λ_{peak}. Since we'll be setting the derivative to zero, we can drop constant coefficients in the expression for the SED.

$$\nu F_\nu \propto \nu B_\nu Q_{emis}(\nu) \propto \nu^2 B_\nu \propto \frac{\nu^5}{e^{\frac{h\nu}{kT_{grain}}} - 1} \tag{15}$$

Define $x \equiv \frac{h\nu}{kT_{grain}}$, take the derivative of equation 15, and set it equal to 0:

$$\frac{d(\nu F_\nu)}{d\nu} \propto \frac{d}{d\nu} \frac{\nu^5}{e^{\frac{h\nu}{kT_{grain}}} - 1} \propto \frac{d}{dx} \frac{x^5}{e^x - 1} = \frac{5x^4(e^x - 1) - x^5 e^x}{(e^x - 1)^2} = 0. \tag{16}$$

Using Mathematica's NSolve on the numerator, we find that $x_{peak} = 4.97$.

$$\nu_{peak} = \frac{kT_{grain}}{h} x_{peak} \Rightarrow \lambda_{peak} = \frac{ch}{kT_{grain} 4.97} = 140\,\mu m \tag{17}$$

(d) These largest grains also carry the lion's share of the mass in the interstellar grain distribution. Given a dust-to-gas mass density ratio of $\rho_{dust}/\rho_{gas} \sim 10^{-2}$ (metallicity), a rough average density of gas in the Galaxy of 0.1 H atom cm^{-3}, and a radial extent of the Galaxy of 50 kpc (kiloparsecs), calculate the specific intensity of the infrared Milky Way at a wavelength of λ_{peak}. Express in mJy arcsec^{-2}, where mJy = milliJansky = $10^{-3} \times 10^{-23}\,\mathrm{erg\,s^{-1}\,cm^{-2}\,Hz^{-1}}$ is a radio astronomer's unit of "flux density" (the "density" here refers to spectral density; i.e., it refers to the per Hertz) and arcsec = 1 arcsecond = 1/206265 (a phone number worth remembering) radians.

Let's again assume we're in the center of the Galaxy, so our line-of-sight distance is r_{Gal}. We can calculate the number density of grains in the Galaxy, using the information given in the problem, and an estimate of the density of a single dust grain ($\rho_{grain} \sim 2\,\mathrm{g\,cm^{-3}}$):

$$n_{dust} = \frac{\rho_{dust}}{\rho_{gas}} n_{gas} m_H (\frac{4}{3}\pi a^3 \rho_{grain})^{-1} = 2 \times 10^{-13} cm^{-3} \tag{18}$$

Now we multiply n_{dust} by r_{Gal} to get a column density, and then multiply by the effective cross-sectional area of a grain, $Q_{emis}\pi a^2$, to obtain the optical depth along the line of sight:

$$\tau_{los} = (n_{dust})(r_{Gal}) Q_{emis}(\pi a^2) = 0.04 \tag{19}$$

These grains comprise a homogeneous, thermally emitting (and absorbing) slab of source function $S_\nu = B_\nu$. The observed specific intensity for such a slab equals

$$\begin{aligned} I_{\nu_{peak}} &= S_{\nu_{peak}}(1 - e^{-\tau_{los}}) && (20)\\ &= B_{\nu_{peak}}(T_{grain})(1 - e^{-0.04}) && (21)\\ &\approx B_{\nu_{peak}}(T_{grain}) \times 0.04 && (22)\\ &\approx 3 \times 10^{-14}\,\mathrm{erg\,s^{-1}\,cm^{-2}\,Hz^{-1}\,sr^{-1}} && (23)\\ &\approx 80\mathrm{mJy\,arcsec^{-2}} && (24) \end{aligned}$$

using $1\,\mathrm{sr} = (206265\,\mathrm{arcsec})^2$. We bravely compare this answer to the truth as measured by the DIRBE sky map (http://lambda.gsfc.nasa.gov/product/cobe/cobe_images/aaf.gif): in the plane of the Galaxy, DIRBE measured ~8000 MJy/sr at $\lambda = 100$ microns. Our answer of 80 mJy/square arcsecond converts to ~3000 MJy/sr at $\lambda = 140$ microns. Not bad for an order-of-magnitude estimate!

Problem 2. Dust Opacity

A rough model for the dust in the ISM tells us $dn/da \propto a^{-3.5}$, where dn is the differential number of dust grains having radii between a and $a + da$. The largest radius in the distribution is $a_{max} = 0.1\mu m$, and the smallest radius is $a_{min} = 0.001\mu m$.

(a) Plot the opacity, $\kappa(\lambda)$, contributed by all dust grains as a function of wavelength from $\lambda = 0.1\mu m$ to $\lambda = 10\mu m$. Express the opacity in units of $\mathrm{cm^2\,g^{-1}}$, where the $\mathrm{g^{-1}}$ equals "per gram of gas." Use whatever parameters you need as given by problem 1 of this set.

Indicate over every decade in wavelength which grain sizes dominate the opacity. (For example, at wavelengths between $\lambda = 0.1$ and 1 μm, do the $a \sim 0.1\mu m$ grains dominate *the opacity? If not, do the $a \sim 0.01\mu m$ grains dominate? And if not them, what about the $a \sim 0.001\mu m$ grains?)*

By definition,

$$\rho_{dust}\kappa_{dust}(\lambda) = \int_{a_{min}}^{a_{max}} \frac{dn}{da} Q_{abs}(a,\lambda)\pi a^2 da \tag{25}$$

where dn/da is the differential number *density* of grains having radii between a and $a + da$, and ρ_{dust} is the volumetric (bulk) density of grains in space (not the internal grain density, ρ_{grain}, which is like that of water). Note also that κ_{dust} is the cross-section for absorption by dust per gram of DUST. At the very end of the problem, we only have to multiply by the dust-to-gas mass ratio, 10^{-2}, to get the desired cross-section per gram of GAS.

Begin by finding an expression for dn/da. All we are told is that $dn/da \propto a^{-3.5}$, so we need to find the constant of proportionality. We can find it by using the fact that the mass of all grains, integrated over the entire size range, must yield a volumetric mass density of ρ_{dust}. In other words, defining $dn/da = Ca^{-3.5}$,

$$\rho_{dust} = \int_{a_{min}}^{a_{max}} \frac{dn}{da}\frac{4}{3}\pi a^3 \rho_{grain} da \tag{26}$$

$$\approx \frac{4\pi}{3} C \rho_{grain} a_{max}^{1/2} \tag{27}$$

$$\rightarrow C = \frac{\rho_{dust}}{\rho_{grain}} \frac{3}{8\pi} a_{max}^{-1/2} \tag{28}$$

since the integral is dominated by the upper limit. Plug C into our beginning expression for κ_{dust}:

$$\kappa_{dust}(\lambda) = \frac{3}{8\rho_{grain}\sqrt{a_{max}}} \int_{a_{min}}^{a_{max}} a^{-1.5} Q_{abs}(\lambda, a) da \tag{29}$$

Now we attack the integral. First we note that for wavelengths longer than $2\pi a_{max} \approx 0.6\,\mu$m, we are always on the falling power-law section of $Q_{abs} = 2\pi a/\lambda$. To wit,

$$\int_{a_{min}}^{a_{max}} a^{-1.5} Q_{abs}(\lambda > 0.6\,\mu\text{m}) da = \frac{2\pi}{\lambda} \int_{a_{min}}^{a_{max}} a^{-0.5} da \tag{30}$$

$$\approx \frac{4\pi}{\lambda}\sqrt{a_{max}} \tag{31}$$

So for $\lambda > 0.6\,\mu$m, plugging our answer for the integral into (29),

$$\kappa_{dust}(\lambda > 0.6\,\mu\mathrm{m}) = \frac{3\pi}{2\rho_{grain}}\lambda^{-1} \tag{32}$$

Finally convert to the desired κ by multiplying by $\rho_{dust}/\rho_{gas} \sim 10^{-2}$:

$$\kappa(\lambda > 0.6\,\mu\mathrm{m}) = \frac{3\pi}{200\rho_{grain}}\lambda^{-1} \tag{33}$$

At these wavelengths, the BIGGEST grains, near a_{max}, dominate the opacity. When we evaluated the integral in (31), a_{max} dominated the integral.

Now we attack $\lambda < 0.6\,\mu$m. The integral in (29) breaks into two pieces,

$$\int_{a_{min}}^{\lambda/2\pi} \frac{2\pi a}{\lambda} a^{-1.5} da + \int_{\lambda/2\pi}^{a_{max}} a^{-1.5} da \tag{34}$$

which evaluates to

$$\frac{4\sqrt{2\pi}}{\sqrt{\lambda}} - \frac{4\pi}{\lambda}\sqrt{a_{min}} - \frac{2}{\sqrt{a_{max}}} \tag{35}$$

This cumbersome expression numerically evaluates to 2141 (cgs) at $\lambda = 0.1\,\mu$m, where the major contribution is from the first (and only positive) term. Therefore at $\lambda = 0.1\,\mu$m, the grains that contribute most to the opacity are those whose radii $a \approx \lambda/2\pi \approx 0.01\,\mu$m. At $\lambda = 0.6\,\mu$m, the cumbersome expression evaluates to 630 (cgs); the major contribution is from $a \approx 0.1\,\mu$m.

Putting it all together, the opacity (per gram of GAS) at $\lambda = 0.1\,\mu$m equals

$$\kappa(\lambda = 0.1\,\mu\mathrm{m}) = 2500\,\mathrm{cm}^2\,\mathrm{g}^{-1} \tag{36}$$

(dominated by smallish grains, $a \approx 0.01\,\mu$m), falling to

$$\kappa(\lambda = 0.6\,\mu\mathrm{m}) = 700\,\mathrm{cm}^2\,\mathrm{g}^{-1} \tag{37}$$

(dominated by the biggest grains, $a \approx 0.1\,\mu$m), and falling from thereon as λ^{-1}, so that at $\lambda = 10\,\mu$m,

$$\kappa(\lambda = 10\,\mu\mathrm{m}) = 42\,\mathrm{cm}^2\,\mathrm{g}^{-1} \tag{38}$$

(and dominated throughout this long wavelength regime by the biggest grains, $a \approx 0.1\,\mu$m).

(b) Convert your plot to read magnitudes of absorption per kiloparsec travelled in the Galaxy (mag/kpc) as a function of λ. Again, use whatever parameters you need from problem 1 above.

A magnitude is a logarithmic unit measured in base 2.5; each magnitude of absorption reduces the flux from a star by a factor of 2.5; a rule of thumb is that 5 magnitudes is a factor of 100. (Editorial note: Astronomer's magnitudes were invented by Hipparcos to measure star brightnesses; a magnitude 0 star is 2.5 times brighter than a magnitude 1 star, and so on. For some reason, optical and near-infrared astronomers refuse to abandon this system, even though it runs backwards, crosses zero, and is normalized to different values depending on the wavelength band. As far as I can see, its only redeeming quality is that base 2.5 is close to the natural base e = 2.712...)

By definition,

$$\tau_\lambda(s) = \rho_{gas}\kappa_\lambda s \tag{39}$$

As light passes through a distance s in the ISM, the flux is reduced by $e^{\tau_\lambda(s)}$. We want to plot this quantity on the magnitude scale, which is base $100^{1/5}$, and which *increases* as flux is *reduced*. Therefore,

$$mag_\lambda(s) = -log_{100^{1/5}} e^{-\tau_{lambda}(s)} = \tau_\lambda(s) log_{100^{1/5}} e = \tau_\lambda(s)\frac{1}{\frac{2}{5}ln10} \tag{40}$$

so

$$\left(\frac{mag}{kpc}\right)_\lambda = 1.086 \times \rho_{gas}\kappa_\lambda \times \frac{3\times10^{21}cm}{kpc} \tag{41}$$

where $\rho_{gas} = n_{gas}m_H$ from 1d. Plugging in numbers from 2a and from 1d, we get

$$\left(\frac{mag}{kpc}\right)_{\lambda=0.1\,\mu\mathrm{m}} = 1.4\,\frac{mag}{kpc} \tag{42}$$

$$\left(\frac{mag}{kpc}\right)_{\lambda=0.6\,\mu\mathrm{m}} = 0.4\,\frac{mag}{kpc} \tag{43}$$

$$\left(\frac{mag}{kpc}\right)_{\lambda=10\,\mu\mathrm{m}} = 0.02\,\frac{mag}{kpc} \tag{44}$$

The plot of mag/kpc (λ) has identical shape to the plot for $\kappa(\lambda)$. Moral: the extinction at infrared wavelengths is much reduced below that in the optical.