Astro 201 – Radiative Processes – Solution Set 9

Joe Converse and Eugene Chiang

NOTE FOR ALL PROBLEMS: In this class, we are not going to stress over the $\sin\alpha$ term in any expression involving synchrotron emission, nor are we going to worry about other factors of order unity, like gamma (Γ) functions. If you see a $\sin\alpha$ or gamma function in your travels, just set it equal to 1.

Problem 1. *Synchrotron Losses*

(a) Obtain an analytic expression for the energy of a single relativistic electron as a function of time, $E(t)$, taking into account its energy loss by synchrotron radiation. Your expression should contain only the variables $E(0)$ (the initial energy of the electron), B (the magnetic field, here held fixed with time, following the rest of the world, though one should worry in general about the field changing with time just as the electron energy spectrum changes with time), time t, and fundamental constants. Assume $\sin\alpha = 1$ (the electron pitch angle is 90 degrees) for simplicity.

For (synchrotron) problems of interest to us, the electron always remains relativistic. It merely evolves from a large $\gamma \gg 1$ to a smaller $\gamma \gg 1$.

We know what the power emitted is:

$$P = \frac{dE}{dt} = -\frac{4}{3}\sigma_T c\gamma^2\beta^2 u_B \tag{1}$$

where the minus is because the energy is being lost. Since $\gamma \gg 1$ always, $\beta \approx 1$ always. So we have

$$\frac{dE}{dt} = -\frac{4}{3}\sigma_T c\gamma^2 u_B \tag{2}$$

where $\gamma = \frac{E}{m_e c^2}$. Solving this differential equation by separation of variables we get

$$\int_{E_0}^{E} \frac{dE}{E^2} = -\frac{4\sigma_T u_B}{3m_e^2c^3}\int_0^t dt \tag{3}$$

where E_0 is the initial energy. Integrating this and multiplying the result by E_0 gives

$$\frac{E_0}{E} - 1 = \frac{t}{\tau}\text{, where} \tag{4}$$

$$\tau = \frac{3m_e^2c^3}{4\sigma_T u_B E_0} = \frac{\gamma_0^2 m_e^2 c^4}{\frac{4}{3}\sigma_T c \gamma_0^2 u_B E_0} = \frac{E_0^2}{P_0 E_0} = \frac{E_0}{P_0} \tag{5}$$

which is the synchrotron lifetime (all subscript 0's refer to the value at $t = 0$). Solving for E we find

$$E(t) = \frac{E_0}{1 + \frac{t}{\tau}} \tag{6}$$

(b) How can you reconcile the loss of energy of the electron with the bald statement of Rybicki & Lightman on page 168 that "γ is constant"?

Rybicki & Lightman say on page 167 that $\frac{\partial}{\partial t}(\gamma m_e c^2) = q\vec{v} \cdot \vec{E} = 0$, and from there conclude that γ=constant because $\vec{E} = 0$. They are referring to the fact that the *external* E-field is 0. The electron, however, also feels its own E-field, and so the total $\vec{E} \neq 0$, and hence γ is not actually constant. It is, however, very nearly constant over 1 gyro-period (except near ultrastrong magnetic fields, $\sim 10^{15}$ G), and the dynamics over 1 gyro-period is all that concerns RL on page 167.

The electron feeling its own E-field is referred to as "radiation reaction." All synchrotron radiation is a consequence of radiation reaction.

(c) We have made arguments in class that power-law distributions of electrons in astrophysical sources are maintained against synchrotron losses by continuous energization by central engines (a.k.a. injection). The injection (input) spectrum of electrons is modified by synchrotron losses to produce a steady-state (output) distribution.

Call $\eta(E,t) = dN/dE$ the differential energy spectrum of electrons as discussed repeatedly in class. Continuity of electrons in energy space reads

$$\frac{\partial \eta(E,t)}{\partial t} + \frac{\partial}{\partial E}[\dot{E}\eta(E,t)] = I \tag{7}$$

where I is the rate of injection of electrons with some input distribution and $\dot{E}$ is the rate of energy loss of a single electron by synchrotron radiation. This equation should not mystify you; it merely describes how the number of electrons in a given energy bin changes with time, taking into account a flux divergence (the second term on the left-hand-side) and a source term (the right-hand-side).

We have assumed in class a steady-state distribution of electron energies for which $\eta \propto E^p$. Given p, how must I scale with E? Give only the scaling and forget about the numerical coefficients.

As with most scaling problems, you don't have to solve anything in detail. Ruthlessly work to order of magnitude.

From the definition of I we have

$$I \propto \dot{E}\frac{\partial \eta}{\partial E} + \eta\frac{\partial \dot{E}}{\partial E} + \dot{E}\frac{\partial \eta}{\partial E} \tag{8}$$

and we know $\eta \propto E^p$, so $\frac{\partial \eta}{\partial E} \propto E^{p-1}$. For $\dot{E}$ we have

$$\dot{E} = -\frac{E_0}{\tau(1+\frac{t}{\tau})^2} \tag{9}$$

and from part (a) we know that $1 + \frac{t}{\tau} = \frac{E_0}{E}$. Then

$$\dot{E} = -\frac{E_0}{\tau}\left(\frac{E}{E_0}\right)^2 = -\frac{E^2}{E_0\tau} \propto E^2 \tag{10}$$

and $\frac{\partial \dot{E}}{\partial E} \propto E$. Therefore

$$I \propto E^2E^{p-1} + E^pE + E^2E^{p-1} \propto E^{p+1} \tag{11}$$

(d) Electrons having a given energy must wait a characteristic time before synchrotron losses become important. Before this time elapses for all such electrons, how does η scale with E?

Before losses become important, $t \ll \tau$, and so $E \approx E_0$ and $\dot{E} \approx -\frac{E_0}{\tau} \propto E$. Then we have

$$I \propto E^{p+1} \propto E\frac{\partial \eta}{\partial E} + \eta \tag{12}$$

which gives that $\eta \propto E^{p+1}$. In other words, it looks just like the injected spectrum.

An easier way to find the same result is to set the second term on the LHS of equation (7) equal to zero, appropriate to the case of zero energy loss. Then $\eta = It + \eta_0$, and if $\eta_0 = 0$, then $\eta \propto I \propto E^{p+1}$. The spectrum looks just like the injected spectrum in slope, but its amplitude rises linearly with time.

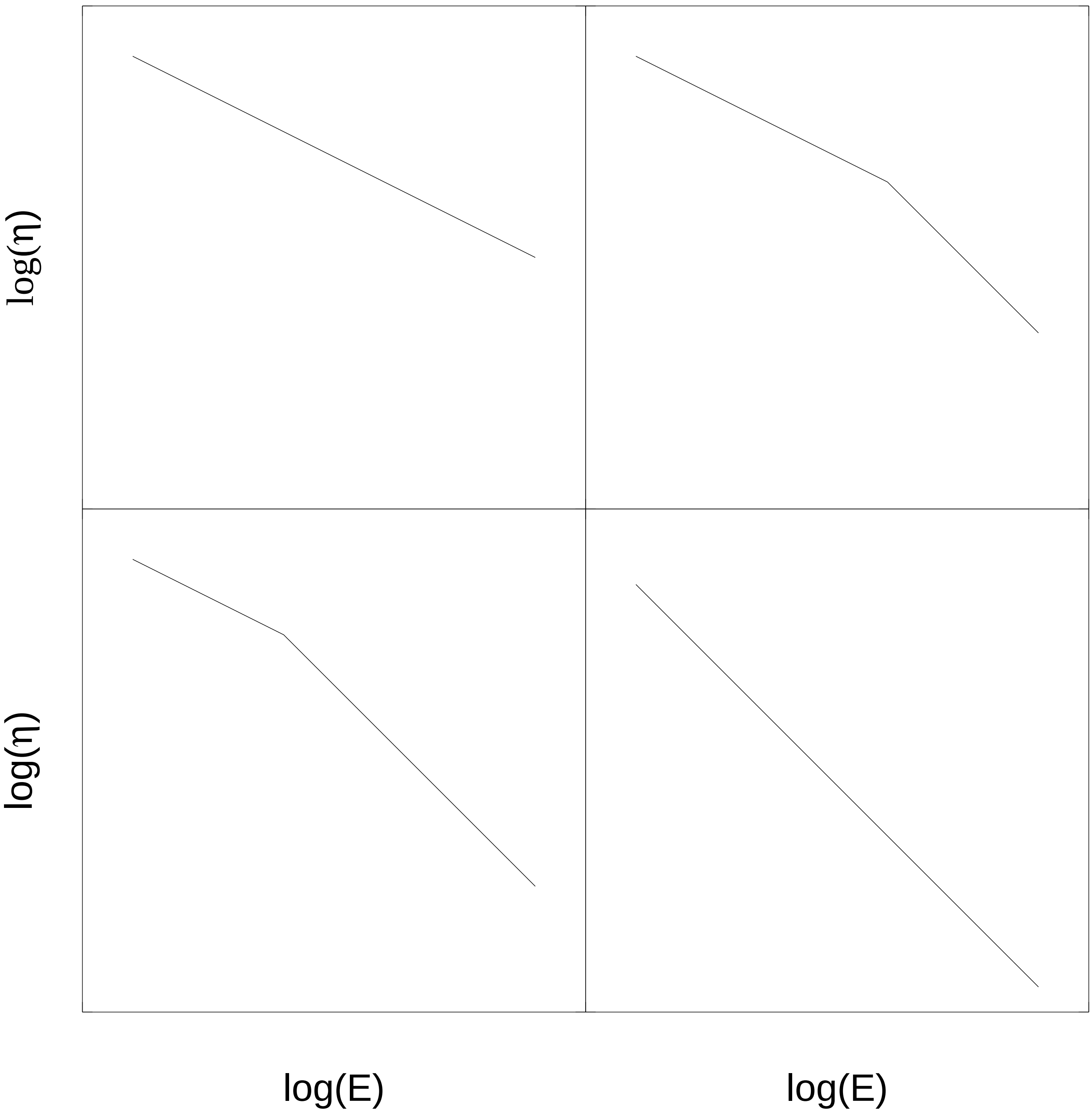
(e) The spectral index ($\alpha = d\ln F_\nu / d\ln \nu$) of radiation in a fixed frequency range from a radio jet flattens with increasing distance from the central galaxy. That is, $\alpha = -0.5$ at the remote edge of the jet (the "hot spot"), and $\alpha = -1$ closer in. Given your

understanding in (c) and (d), where are the "freshest" electrons located, i.e., those newly injected into the energy spectrum? Are they at the end of the jet, or are they closer in? In other words, where is the principal site of particle acceleration?

Since $\alpha = \frac{1+p}{2}$, for the hot spot ($\alpha = -0.5$), $p = -2$, and farther in on the jet ($\alpha = -1$), $p = -3$. Hence the hot-spot (where the spectrum is flatter) looks more like the injected spectrum, and so it is where the "young" elections are. It is when the jet is stopped at the lobe that most of the acceleration occurs.

(f) Sketch several profiles of η vs. E at various times, assuming I is constant in time.

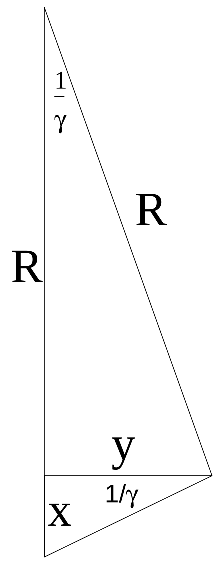
Since the synchrotron lifetime goes as E_0^{-1}, the higher-energy electrons lose their energy first. Initially the spectrum is $\propto E^{p+1}$ (upper-left panel), but as time goes on, the high-energy end steepens to be $\propto E^p$ (upper-right panel). The point of steepening moves to lower energies as time goes on (lower-left panel), until eventually the entire spectrum reaches the steady state of being $\propto E^p$ (lower-right panel).



Problem 2. *Great Balls of Relativistic Fire*

Rybicki & Lightman Problem 4.1 (Don't worry if you can't reproduce the answer to within a factor of 2; the answer is meant to be a rule of thumb.)

Since it is optically thick, we will only see light from the nearest side. Even then, because of beaming, only those parts of the object within an angle of about $\frac{1}{\gamma}$ will send a signal to us. This gives the geometry shown in the figure below.



The observed time delay is $c\Delta t > x$. The geometry gives us $y = R \sin \frac{1}{\gamma} \approx \frac{R}{\gamma}$. Then we see that $x = y \tan \frac{1}{\gamma} \approx \frac{R}{\gamma}\frac{1}{\gamma} = \frac{R}{\gamma^2}$. So therefore

$$R < \gamma^2 c\Delta t \tag{13}$$

Problem 3. *Superluminal Motion, or Photons Chasing Photons*

Rybicki & Lightman Problem 4.7

The geometry for this problem is very similar to that of problem 2. In this case, in a time dt, the light coming straight down at us travels a distance of cdt, while the material going along the jet at an angle θ (instead of $\frac{1}{\gamma}$ above) travels vdt. The change in apparent position on the sky is $\Delta x = vdt \sin\theta$, and the distance it traveled parallel to our line of sight is $vdt \cos\theta$. The apparent difference in distance between the original light pulse and one emitted after dt is $cdt - vdt\cos\theta = c\Delta t$. Finally,

$$v_{app} = \frac{\Delta x}{\Delta t} = \frac{v dt \sin\theta}{dt - \frac{v dt}{c}\cos\theta} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta} \tag{14}$$

For $v = 0.99c$ and $\theta = 0.1$, $v_{app} = 6.6c$, and hence it can appear to move superluminally. The maximum apparent speed occurs when

$$0 = \frac{dv_{app}}{d\theta} = \frac{v\cos\theta - \frac{v^2}{c}}{(1 - \frac{v}{c}\cos\theta)^2} \tag{15}$$

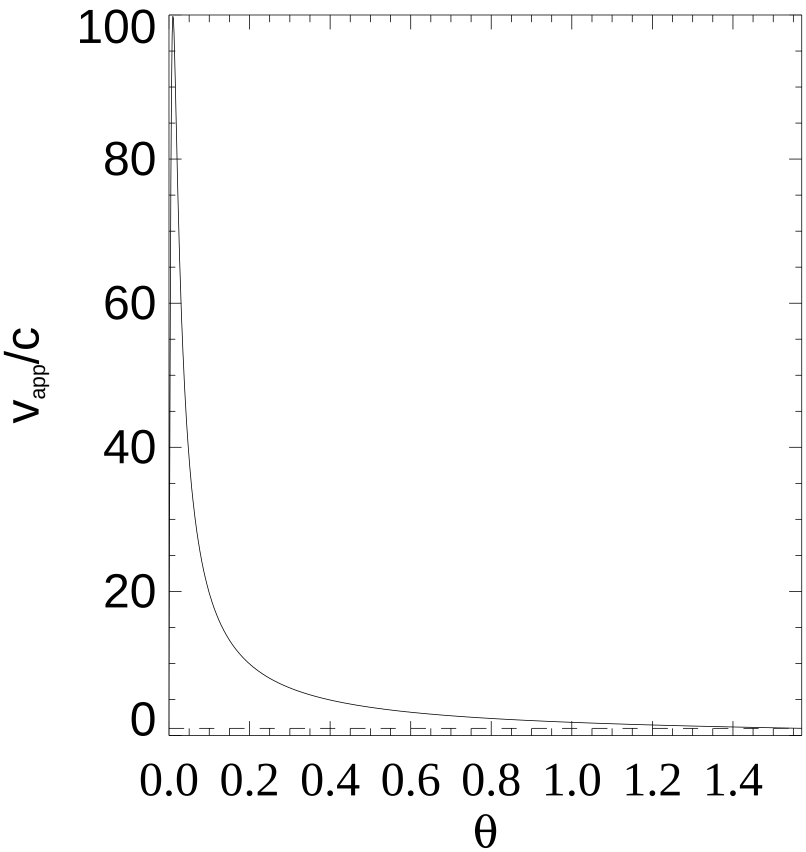
which is true when $\cos\theta = \frac{v}{c}$. At this angle, $\sin\theta = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma}$, and

$$v_{app,max} = \frac{\frac{v}{\gamma}}{1 - \frac{v}{c}\frac{v}{c}} = \frac{v\gamma^2}{\gamma} = \gamma v \tag{16}$$

(c) Plot v_{app}/c vs. θ for $\gamma = 10^2$. Does the viewing angle θ need to be especially small for superluminal motion to be perceived?

For culture: see the beautiful illustration of superluminal motion in the optical M87 jet by Biretta in the accompanying .jpg on the class website.

For $\gamma = 10^2$, $v = 0.99995c$. As the plot below shows, almost all viewing angles below $\theta = \frac{\pi}{2}$ exhibit superluminal motion if the jet is moving fast enough (dotted line is $\frac{v_{app}}{c} = 1$).


100
80
60
40
20
0
v_{app}/c
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
θ