1. **Radiative Diffusion and the Greenhouse Effect**

The Earth’s atmosphere is relatively optically thin at the visible wavelengths emitted by the sun. As a result most of the sunlight incident on the Earth makes it through the Earth’s atmosphere. But the radiation emitted by the surface of the Earth isn’t in the visible, because the temperature of the surface is quite cold. Instead, the radiation is in the IR. And the opacity of the Earth’s atmosphere at infrared wavelengths is much larger than at visible wavelengths because of absorption by water vapor, carbon dioxide, chlorofluorocarbons, ... As a result not all of the infrared emission generated by the Earth actually escapes immediately. Some of it is trapped, increasing the temperature at the surface of the Earth. This is the greenhouse effect, which we will quantitatively model in this problem.

Consider an opaque object/atmosphere that has a net flux $F$ going through it. The energy is carried by radiation. The temperature at the “bottom” (the interior, close to the central star or planet) is $T_{in}$ and the temperature at the “top” (the exterior, further from the star or planet) is $T_{ext}$. It helps to draw a picture of this.

a) Argue that the net flux $F$ can be written as $F = \sigma T_{ext}^4$. Why is $F = \sigma T_{in}^4$ incorrect?

b) Use the radiative diffusion equation to show that $T_{in} \sim \tau^{1/4} T_{ext}$ where $\tau = \rho \kappa \Delta R$ is the optical depth through the atmosphere and $\Delta R$ is its thickness.

This is an extremely useful result that relates the interior and exterior temperatures to the total optical depth.

c) Use the result from b) to estimate the central temperature of the sun (given the observed surface temperature, mass, and radius).

d) In the absence of the greenhouse effect (i.e., ignoring the fact that the IR emission emitted by the planet can’t easily get through the atmosphere), calculate the temperature of the surface of the planet (let’s call it $T_{nogreen}$).

Now assume that the planet has an atmosphere that is optically thin to visible light (except for the albedo $A$) but is optically thick to infrared emission. The atmosphere is like that described in the first part of this problem.

e) By considering the total energy per unit time entering and leaving the atmosphere of the planet, show that the temperature at the top of the atmosphere must satisfy $T_{ext} = T_{nogreen}$. Use this result and your result in b) to estimate the temperature in the inner part of the planet’s atmosphere $T_{in}$. This is an estimate of the surface temperature of the planet taking into account the greenhouse effect.

f) Estimate the optical depth of the Earth’s atmosphere to infrared radiation by comparing the temperature you get using e) with the true surface temperature of the Earth. The Albedo of the Earth is 0.3.

g) Venus is the extreme example of the greenhouse effect in the solar system. The surface temperature of Venus is $\approx 740$ K (hot enough to melt lead!). In the absence of the greenhouse effect what would Venus’s temperature be? ($a = 0.72$ AU for Venus and $A = 0.7$). In reality, Venus has an incredibly opaque atmosphere in the infrared, with $\tau \approx 70$. What does the model above predict for its surface temperature?1

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1The model does reasonably well, but not great.
2. The surface of a star (the “photosphere”) is the place where the mean free path of the photons $\ell$ is comparable to the scale-height $h$ of the atmosphere (as we will discuss in class). At smaller radii (deeper in the star), the density is higher and $\ell \ll h$, which implies that the photons bounce around many times; at larger radii $\rho$ is smaller, $\ell \gg h$, and the photons are rarely absorbed and so travel on straight lines to us. Thus $\ell \approx h$ is a good approximation to the place in the atmosphere of a star where most of the light we see originates.

a) The temperature at the photosphere of the sun is 5800 K. Estimate the mass density $\rho$ in the photosphere. Assume that Thompson scattering dominates the opacity.

b) In reality, the temperature of the surface of the sun is so low that hydrogen is primarily neutral. There are thus not enough free electrons for Thompson scattering to be important. The opacity at the surface of the sun is instead due to the negatively charged ion of Hydrogen called the $H^-$ ion; the opacity due to $H^-$ is given by $\kappa \approx 2.5 \times 10^{-31} \rho^{1/2} T^9$ cm$^2$ g$^{-1}$. Using this (correct) opacity, repeat the estimate from a) of the density at the photosphere of the sun.

c) Just beneath the photosphere, energy is transported by convection, not radiation, for the reasons discussed in class (in fact, the photosphere is the place where photons travel so freely out of the star that energy transport by radiation finally dominates over convection). Estimate the convective velocity near the photosphere given your density from b).

d) What is the characteristic timescale for convective “blobs” to move around near the photosphere? How does this compare to the observed timescale for granulation on the surface of the sun, which was a few min in the movie we watched in class?

e) Is the assumption $ds/dr \approx 0$ valid near the surface of the sun? Why or why not?

3. Convective Atmospheres In HW 2, you calculated the structure of a stellar atmosphere in which energy is transported by radiative diffusion; you showed that such an atmosphere satisfies $P \propto \rho^{4/3}$. Here we will consider the problem of a convective atmosphere, which is much more relevant to sun-like stars. For simplicity, assume that the atmosphere is composed of fully ionized hydrogen. The solar convection zone contains very little mass (only $\approx 2\%$ of the mass of the sun). Thus, let’s consider a model in which we neglect the mass of the convection zone in comparison to the rest of the sun. For reasons we will discuss in class, we can model the convection zone as having $P = K \rho^\gamma$ with $\gamma = 5/3$ and $K$ a constant. Let $R_c$ be the radius of the base of the convection zone.

a) Solve for the density, temperature, and pressure as a function of radius in the convection zone. Do not assume that the convection zone is thin (i.e., even though $M_r = \text{constant} = M$, because $r$ changes significantly in the convection zone, do not assume that the gravitational acceleration is constant).

b) In detailed solar models, the pressure at the base of the convection zone is $\approx 5.2 \times 10^{13}$ dyne/cm$^2$ and the density is $\rho \approx 0.175$ g cm$^{-3}$. Using your solution from a), estimate the radius of the base of the convection zone $R_c$. Compare this to the correct answer of $R_c \approx 0.71 R_\odot$.

c) In your model, what is the temperature of the sun at $0.99 R_\odot$, $0.9 R_\odot$, and at the base of the solar convection zone. This gives you a good sense of how quickly the temperature rises from its surface value of $\approx 5800$ K as one enters the interior of the sun.