Homework #6

1. a) What is the classical distance of closest approach for two protons with an energy of 2 keV (the mean thermal energy at the center of the sun)? Estimate the probability that the protons tunnel through the Coulomb barrier trying to keep them apart. Answer the same two questions for two $^4\text{He}$ nuclei and for a proton and a $^4\text{He}$ nucleus with the same energy of 2 keV.

b) What energy $E$ would be required for i) the two $^4\text{He}$ nuclei, ii) the proton and the $^4\text{He}$ nucleus, and iii) two $^{12}\text{C}$ nuclei to have the same probability of penetrating the Coulomb barrier as the two protons? For particles with energies equal to the mean thermal energy of the plasma, what temperatures do these correspond to?

2. Consider two interacting particles with relative kinetic energy equal to $E_0$, the energy, derived in class, of particles that dominate fusion reactions in stars.

i) At a temperature $T$, roughly what fraction of particles in a MB distribution have energies within $dE \sim E_0$ of $E_0$? What is this fraction for protons at the center of the sun?

ii) At the center of the sun, what is the mean free path ($R_\odot$) and mean free time (in yrs) of protons undergoing the reaction $p + p \rightarrow ^2\text{H} + e^+ + \nu_e$? The $S$ value for this reaction is $S = 3.78 \times 10^{-22}$ keV barn.

3. In lecture, we showed that the net rate of nuclear reactions in stars depends on a competition between the MB distribution of velocities/energies, which favors low energies $\sim kT$, and the tunneling probability that favors high energies $\sim E_G$. Quantitatively, the competition was described by the following integral

$$I = \int_0^\infty dE \exp[-f(E)]$$

where $f(E) = E/kT + (E_G/E)^{1/2}$. I claimed in lecture that

$$I \approx \frac{\sqrt{2\pi} \exp[-f(E_0)]}{\sqrt{f''(E_0)}}$$

when we approximate the function $f(E)$ using a Taylor series at the energy $E_0$ (the energy which dominates nuclear reactions). Prove that the integral $I$ is indeed given by that above.

4. The derivation of tunneling in class used the WKB approximation, which requires that the wavelength of the solution is small compared to the length-scale over which the potential changes. Show that this approximation is well-satisfied if the energy of the particles $E$ is significantly smaller than the Gamow energy $E_G$. Recall that most of the contribution to the tunneling happens just inside the classical turning point $r_c$.

5. Consider hydrogen fusion by the CNO cycle illustrated in Fig. 4.5 of Phillips. We will discuss this extensively in class, but this problem is self-contained and you should be able to do this problem before we get to the CNO cycle in class. The slowest reaction in the CNO cycle is $p + ^{14}\text{N} \rightarrow ^{15}\text{O} + \gamma$ for $T \sim 10^7$ K.

a) Calculate the temperature dependence of the CNO cycle (i.e., $\beta$ in $\epsilon \propto T^\beta$) at $1.5 \times 10^7$ K and $3 \times 10^7$ K. To do this, note that the slowest reaction in a cycle/chain of reactions sets the timescale and thus the energy generation rate $\epsilon$ for the entire cycle/chain.

b) It is believed that approximately 1.6% of the solar luminosity is produced by the CNO cycle, with the rest produced by proton-proton fusion. What fraction of the solar luminosity would be produced by the CNO cycle if the central temperature of the sun were 10% larger than its current temperature of $\approx 1.5 \times 10^7$ K?

c) Use the temperature dependence in a) to explain qualitatively why all massive stars have nearly the same central temperature.