Homework #7

1. Imagine that the “S-value” for the first step in the p-p chain, \( p + p \rightarrow 2H + e^+ + \nu_e \), was \( 10^{-10} \) keV barn instead of \( 3.78 \times 10^{-22} \) keV barn. Also assume that the luminosity of the sun would be unchanged (this is not exact, but is not unreasonable since, as you know, the luminosity is largely set by the rate at which photons leak out, not by nuclear fusion!).

   a) Explain why pp fusion would dominate over the CNO cycle in this hypothetical new sun.

   b) Assume the \( p + p \) step in the pp chain is still the rate limiting step. Estimate the central temperature \( T_c \) (in K), radius \( R \) (in \( R_\odot \)), and effective temperature of the new sun.

2. The Main Sequence for Fully Convective Stars (20 pts): In this problem we will determine the main sequence for fully convective low mass stars. We showed in lecture that fully convective stars have \( T_{\text{eff}} \approx 4000 (L/L_\odot)^{1/102} (M/M_\odot)^{7/51} \text{ K} \) (I actually derived a coefficient of 3000 K in lecture but commented that more detailed calculations get something similar but with the coefficient closer to the value of 4000 K used here). We can also write this result as

\[
L \approx 0.2 (M/M_\odot)^{4/7} (R/R_\odot)^{2} L_\odot \equiv L_{\text{conv}}
\]

I called this luminosity \( L_{\text{conv}} \) since it is derived from the properties of energy transport alone (convective interior + radiative atmosphere with H\(^{-}\) opacity). The luminosity of a star is also given by

\[
L_{\text{fusion}} = \int 4\pi r^2 \rho \epsilon(T, \rho) dr
\]  

(1)

where \( \epsilon \) is due to the proton-proton chain for low mass stars (this was given in lecture). As discussed in class, the main sequence is determined by the requirement that the energy escaping the star (in this case by convection) is equal to the energy generated in the star (in this case by pp fusion), i.e., that \( L_{\text{conv}} = L_{\text{fusion}} \).

   a) Use scaling arguments to derive the power-law relations \( R(M), L(M), T_c(M), \) and \( L(T_{\text{eff}}) \) (the HR diagram) for fully convective stars, like we did for other examples in lecture. Approximate \( \epsilon \propto \rho T^\beta \) with an appropriate choice of \( \beta \) (recall that low mass stars will have somewhat lower central temperatures than the sun, closer to \( \approx 5 \times 10^6 \) K, as you will see in part b).

   In a) you just determined a scaling relation between stars of different mass, but not the absolute values of \( L, T_{\text{eff}}, \) etc. In class, we did the latter by scaling to the sun. Note, however, that it is not reasonable to estimate the properties of low mass stars by scaling from the properties of the sun, since the sun is not a fully convective star! Instead we need to actually determine the structure of some fully convective star. This is what we will do in the rest of the problem.

   We can significantly improve on the above scaling arguments by using the fact that fully convective stars are \( n = 3/2 \) polytropes. It turns out that for a polytrope, \( \epsilon \) in equation (1) can be Taylor expanded near the center to yield

\[
L_{\text{fusion}} \simeq \frac{2.4 \epsilon_c M}{(3 + \beta)^{3/2}}
\]

(2)

where I have again approximated \( \epsilon \propto \rho T^\beta \) and where \( \epsilon_c \) is \( \epsilon \) evaluated at the center of the star. I am not asking you prove equation (2). You will have to trust me. Note that for a typical value of \( \beta \) for the pp chain, equation (2) says that \( L_{\text{fusion}} \approx 0.1 \epsilon_c M \). This makes sense because fusion only takes place at the center of the star (not all of the mass participates).

   b) Use the results for \( n = 3/2 \) polytropes from HW 4, Problem # 3, to write the central temperature of the star \( T_c \), central density \( \rho_c \), and pp energy generation at the center of the star \( \epsilon_c \) in terms of the mass \( M \) and radius \( R \). Assume \( X = 0.7 \) and \( \mu = 0.6 \) (typical for stars
just reaching the main sequence). Note that you should give expressions for $T_c$, $\rho_c$, and $\epsilon_c$ here, with constants and real units, not just scaling relationships. So that the constants in front of your expressions are reasonable, please normalize $M$ to $M_\odot$ and $R$ to $R_\odot$.

c) Use equation (2), the results of b), and $L_{\text{conv}} = L_{\text{fusion}}$ on the main sequence to determine the $R(M)$, $L(M)$, $T_c(M)$, and $L(T_{\text{eff}})$ relations for fully convective stars. If you use the same $\beta$, your expressions here should be the same as in a) except that you should now be able to determine the absolute normalization for $R(M)$, $L(M)$, etc., i.e., you have determined the true luminosity and radius of a fully convective star from first principles. In doing this problem, remember that $\beta$ is temperature dependent so make sure you check that your value of $\beta$ is reasonable given the resulting central temperature that you calculate.\footnote{The right way to do this is to do an iteration where you guess a $\beta$, find $T_c$, adjust $\beta$ to be appropriate for the new $T_c$, re-calculate $T_c$, etc. You don’t have to do something that fancy. But do pick a reasonable $\beta$ appropriate for colder low mass stars, not one that is appropriate at $10^8$ K!}

d) What are your predicted luminosities, radii, and effective temperatures for main sequence stars with $M = 0.1$ and $0.3M_\odot$? Compare your values to the values of $L = 0.01L_\odot$, $R = 0.3R_\odot$, and $T_{\text{eff}} = 3450$ K for $M = 0.3M_\odot$ and $L = 10^{-3}L_\odot$, $R = 0.11R_\odot$, and $T_{\text{eff}} = 3000$ K for $M = 0.1M_\odot$ that I found in a graduate textbook (based on detailed models).

3. Very Massive Stars: Consider very massive stars with $M \sim 50-100M_\odot$. Recall that in such stars, radiation pressure due to photons (a relativistic particle) is more important than gas pressure. Fusion is by the CNO cycle. Assume for now that energy is transported primarily by photons and that the opacity is due to Thomson scattering (reasonable for hot massive stars).

a) Use scaling arguments to derive the power-law relations $R(M)$, $L(M)$, $T_c(M)$, and $L(T_{\text{eff}})$ (the HR diagram) for very massive stars (as in the previous problem).

b) Estimate the fraction of the mass in the star that is undergoing convection (recall that fusion by the CNO cycle is very concentrated at small radii because of the strong temperature dependence). For comparison, detailed calculations show that the fraction of the mass that undergoes “core” convection increases from $\approx 10\%$ at $2M_\odot$ to $\approx 75\%$ at $60M_\odot$.

c) Calculate the main sequence lifetime of a very massive star as a function of its mass $M$. 
